A NOTE ON THE EXISTENCE OF SOLUTIONS WITH PRESCRIBED ASYMPTOTIC BEHAVIOR FOR HALF-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Manabu Naito, Matsuyama

Received November 25, 2022. Published online July 13, 2023.
Communicated by Alexandr Lomtatidze

Abstract. The half-linear differential equation

$$(|u'|^{\alpha}\operatorname{sgn} u')' = \alpha(\lambda^{\alpha+1} + b(t))|u|^{\alpha}\operatorname{sgn} u, \quad t \geqslant t_0,$$

is considered, where α and λ are positive constants and $b(t)$ is a real-valued continuous function on $[t_0, \infty)$. It is proved that, under a mild integral smallness condition of $b(t)$ which is weaker than the absolutely integrable condition of $b(t)$, the above equation has a nonoscillatory solution $u_0(t)$ such that $u_0(t) \sim \mathrm{e}^{-\lambda t}$ and $u_0'(t) \sim -\lambda \mathrm{e}^{-\lambda t}$ $(t \to \infty)$, and a nonoscillatory solution $u_1(t)$ such that $u_1(t) \sim \mathrm{e}^{\lambda t}$ and $u_1'(t) \sim \lambda \mathrm{e}^{\lambda t}$ $(t \to \infty)$.

Keywords: half-linear differential equation; nonoscillatory solution; asymptotic form

MSC 2020: 34D05, 34D10, 34C11

1. Introduction

In this paper we consider the half-linear ordinary differential equation

$$(|u'|^{\alpha}\operatorname{sgn} u')' = \alpha(\lambda^{\alpha+1} + b(t))|u|^{\alpha}\operatorname{sgn} u, \quad t \geqslant t_0, \tag{1.1}$$

where $\alpha > 0$ and $\lambda > 0$ are constants and $b(t)$ is a real-valued continuous function on $[t_0, \infty)$. If $\alpha = 1$, then (1.1) reduces to the linear equation

$$u'' = (\lambda^2 + b(t))u, \quad t \geqslant t_0. \tag{1.2}$$

It is known that basic results and qualitative results for the linear equation (1.2) can be generalized to the half-linear equation (1.1). The important works relating to (1.1) are summarized in the book of Došlý and Řehák (see [2]).

DOI: 10.21136/MB.2023.0158-22

It is well-known (see, for example, [1], page 90 and [3], Corollary 9.2, Chapter XI) that if

$$\int_{t_0}^{\infty} |b(s)|\,\mathrm{d}s < \infty, \tag{1.3}$$

then the linear equation (1.2) has a nonoscillatory solution $u_0(t)$ such that

$$u_0(t) \sim \mathrm{e}^{-\lambda t}, \quad u_0'(t) \sim -\lambda \mathrm{e}^{-\lambda t}, \quad t \to \infty, \tag{1.4}$$

and a nonoscillatory solution $u_1(t)$ such that

$$u_1(t) \sim \mathrm{e}^{\lambda t}, \quad u_1'(t) \sim \lambda \mathrm{e}^{\lambda t}, \quad t \to \infty. \tag{1.5}$$

Recently Naito and Usami in [11], Theorem 1.1 (i) have generalized the above result for the linear equation (1.2) to the half-linear equation (1.1): if (1.3) holds, then the half-linear equation (1.1) has a nonoscillatory solution $u_0(t)$ satisfying (1.4) and a nonoscillatory solution $u_1(t)$ satisfying (1.5).

For the linear equation (1.2) it is also known (see [3], Corollary 9.2, Chapter XI) that if

$$\lim_{t\to\infty} \int_{t_0}^{t} \mathrm{e}^{-2\lambda r} b(r)\,\mathrm{d}r = \int_{t_0}^{\infty} \mathrm{e}^{-2\lambda r} b(r)\,\mathrm{d}r \quad \text{exists and is finite} \tag{1.6}$$

and

$$\int_{t_0}^{\infty} \mathrm{e}^{2\lambda s} \sup_{\sigma \geqslant s} \left| \int_{\sigma}^{\infty} \mathrm{e}^{-2\lambda r} b(r)\,\mathrm{d}r \right| \mathrm{d}s < \infty, \tag{1.7}$$

then (1.2) has a solution $u_0(t)$ satisfying (1.4) and a solution $u_1(t)$ satisfying (1.5). It is easy to see that if (1.3) holds, then (1.6) and (1.7) hold.

In the present paper we consider the half-linear equation (1.1) under the conditions

$$\lim_{t\to\infty} \int_{t_0}^{t} \mathrm{e}^{-(\alpha+1)\lambda r} b(r)\,\mathrm{d}r = \int_{t_0}^{\infty} \mathrm{e}^{-(\alpha+1)\lambda r} b(r)\,\mathrm{d}r \quad \text{exists and is finite} \tag{1.8}$$

and

$$\lim_{t\to\infty} \mathrm{e}^{(\alpha+1)\lambda t} \Phi_\lambda(t) = 0 \quad \text{and} \quad \int_{t_0}^{\infty} \mathrm{e}^{(\alpha+1)\lambda s} |\Phi_\lambda(s)|\,\mathrm{d}s < \infty, \tag{1.9}$$

where

$$\Phi_\lambda(t) = \int_{t}^{\infty} \mathrm{e}^{-(\alpha+1)\lambda s} b(s)\,\mathrm{d}s, \quad t \geqslant t_0. \tag{1.10}$$

It can be checked that if (1.3) holds, then (1.8) and (1.9) hold. Throughout the paper the condition (1.9) is used under the assumption that (1.8) holds.

Let $\alpha = 1$. Then (1.8) becomes (1.6). It is seen that (1.7) implies (1.9) with $\alpha = 1$. In fact, if (1.7) holds, then

$$\lim_{t\to\infty} \int_{t-1}^{t} \mathrm{e}^{2\lambda s} \sup_{\sigma \geqslant s} \left| \int_{\sigma}^{\infty} \mathrm{e}^{-2\lambda r} b(r)\,\mathrm{d}r \right| \mathrm{d}s = 0.$$

Therefore, since

$$\begin{aligned}\int_{t-1}^{t} \mathrm{e}^{2\lambda s} \sup_{\sigma \geqslant s} \left| \int_{\sigma}^{\infty} \mathrm{e}^{-2\lambda r} b(r)\,\mathrm{d}r \right| \mathrm{d}s &\geqslant \int_{t-1}^{t} \mathrm{e}^{2\lambda s}\,\mathrm{d}s \sup_{\sigma \geqslant t} \left| \int_{\sigma}^{\infty} \mathrm{e}^{-2\lambda r} b(r)\,\mathrm{d}r \right| \\ &\geqslant \frac{1}{2\lambda}(1-\mathrm{e}^{-2\lambda})\mathrm{e}^{2\lambda t} \left| \int_{t}^{\infty} \mathrm{e}^{-2\lambda r} b(r)\,\mathrm{d}r \right|,\end{aligned}$$

we find that

$$\lim_{t\to\infty} \mathrm{e}^{2\lambda t} \left| \int_{t}^{\infty} \mathrm{e}^{-2\lambda r} b(r)\,\mathrm{d}r \right| = 0.$$

This implies that if (1.7) holds, then the former half of (1.9) with $\alpha = 1$ holds. It is clear that if (1.7) holds, then the latter half of (1.9) with $\alpha = 1$ holds.

In the next section it is proved that the condition (1.9) is equivalent to the condition

$$\lim_{t\to\infty} \mathrm{e}^{-(\alpha+1)\lambda t} \Psi_\lambda(t) = 0 \quad \text{and} \quad \int_{t_0}^{\infty} \mathrm{e}^{-(\alpha+1)\lambda s} |\Psi_\lambda(s)|\,\mathrm{d}s < \infty, \tag{1.11}$$

where

$$\Psi_\lambda(t) = \int_{t_0}^{t} \mathrm{e}^{(\alpha+1)\lambda s} b(s)\,\mathrm{d}s, \quad t \geqslant t_0. \tag{1.12}$$

We can show the following theorem.

Theorem 1.1. *Suppose that* (1.9) (*or, equivalently,* (1.11)) *holds. Then the half-linear equation* (1.1) *has a solution* $u_0(t)$ *satisfying* (1.4) *and a solution* $u_1(t)$ *satisfying* (1.5).

Theorem 1.1 generalizes the classical result for (1.2) in Hartman [3], Corollary 9.2, Chapter XI and the recent result for (1.1) by Naito and Usami [11], Theorem 1.1 (i).

Consider now the general half-linear equation of the form

$$(|u'|^{\alpha} \operatorname{sgn} u')' = q(t)|u|^{\alpha} \operatorname{sgn} u, \quad t \geqslant t_0, \tag{1.13}$$

where $q(t)$ is a real-valued continuous function on $[t_0, \infty)$. It is well-known (see, for example, [2], Theorem 1.1.1) that all local solutions of (1.13) can be continued to t_0 and ∞, and so all solutions of (1.13) exist on the entire interval $[t_0, \infty)$. An analogue of Sturm's separation theorem remains valid for (1.13) (see, for example, [2], Theorem 1.2.3). Hence, if the equation (1.13) has a nonoscillatory solution, then any other nontrivial solution is also nonoscillatory. Clearly, if $u(t)$ is a solution of (1.13), then, for any constant c, the function $cu(t)$ is also a solution of (1.13). In particular, if $u(t)$ is a solution of (1.13), then so is $-u(t)$. Therefore we can suppose without loss of generality that a nonoscillatory solution of (1.13) is eventually positive. Note that a solution $u_0(t)$ satisfying (1.4) and a solution $u_1(t)$ satisfying (1.5) are eventually positive.

In the last three decades, many results have been obtained in the theory of oscillatory and asymptotic behavior of solutions of half-linear differential equations. It is known that basic results for the second order linear equations can be generalized to the second order half-linear equations. The important works are summarized in the book of Došlý and Řehák (see [2]). For the recent results to half-linear equations we refer the reader to [4]–[15].

This paper is organized as follows. In Section 2 we state several preliminary results which are related to the condition (1.9) and the condition (1.11). The main idea of the proof of Theorem 1.1 is to show the existence of solutions with suitable asymptotic conditions of generalized Riccati differential equations which are associated with the half-linear equation (1.1). A few results concerning generalized Riccati differential equations associated with (1.1) are stated in Section 3. The proof of Theorem 1.1 is presented in Section 4. An example illustrating the main result is provided in Section 5. The Riccati technique is particularly useful in the qualitative theory of half-linear equations, see, for example, [2], [4]–[6], [8]–[15].

2. Preliminary results

In this section we first prove that the condition (1.9) is equivalent to the condition (1.11).

Lemma 2.1. *The condition* (1.9) *holds if and only if the condition* (1.11) *holds.*

To shorten notation, we put

$$\beta = (\alpha + 1)\lambda. \tag{2.1}$$

This notation is used throughout the paper.

P r o o f of Lemma 2.1. Suppose first that (1.9) holds. Integration by parts gives

$$\begin{aligned}\Psi_\lambda(t) &= \int_{t_0}^t \mathrm{e}^{(\alpha+1)\lambda s} b(s)\,\mathrm{d}s = \int_{t_0}^t \mathrm{e}^{2\beta s}\mathrm{e}^{-\beta s} b(s)\,\mathrm{d}s\\ &= -\mathrm{e}^{2\beta t}\Phi_\lambda(t) + \mathrm{e}^{2\beta t_0}\Phi_\lambda(t_0) + 2\beta\int_{t_0}^t \mathrm{e}^{2\beta s}\Phi_\lambda(s)\,\mathrm{d}s\end{aligned}$$

and so

$$\mathrm{e}^{-\beta t}|\Psi_\lambda(t)| \leqslant \mathrm{e}^{\beta t}|\Phi_\lambda(t)| + \mathrm{e}^{2\beta t_0}|\Phi_\lambda(t_0)|\mathrm{e}^{-\beta t} + \frac{2\beta}{\mathrm{e}^{\beta t}}\int_{t_0}^t \mathrm{e}^{2\beta s}|\Phi_\lambda(s)|\,\mathrm{d}s \tag{2.2}$$

for $t \geqslant t_0$. By the former half of the condition (1.9), the first term of the right-hand side of (2.2) tends to 0 as $t \to \infty$. It is clear that the second term of the right-hand side of (2.2) tends to 0 as $t \to \infty$. The last term of the right-hand side of (2.2) also tends to 0 ($t \to \infty$). In fact, by l'Hospital's rule we have

$$\lim_{t\to\infty} \frac{1}{\mathrm{e}^{\beta t}}\int_{t_0}^t \mathrm{e}^{2\beta s}|\Phi_\lambda(s)|\,\mathrm{d}s = \lim_{t\to\infty}\frac{1}{\beta\mathrm{e}^{\beta t}}\mathrm{e}^{2\beta t}|\Phi_\lambda(t)| = \frac{1}{\beta}\lim_{t\to\infty}\mathrm{e}^{\beta t}|\Phi_\lambda(t)| = 0.$$

Therefore we obtain $\lim \mathrm{e}^{-\beta t}\Psi_\lambda(t) = 0$ as $t \to \infty$.

From (2.2) it follows that

$$\begin{aligned}\int_{t_0}^t \mathrm{e}^{-\beta s}|\Psi_\lambda(s)|\,\mathrm{d}s \leqslant{}& \int_{t_0}^t \mathrm{e}^{\beta s}|\Phi_\lambda(s)|\,\mathrm{d}s + \mathrm{e}^{2\beta t_0}|\Phi_\lambda(t_0)|\int_{t_0}^t \mathrm{e}^{-\beta s}\,\mathrm{d}s\\ &+ 2\beta\int_{t_0}^t \mathrm{e}^{-\beta s}\Bigl(\int_{t_0}^s \mathrm{e}^{2\beta r}|\Phi_\lambda(r)|\,\mathrm{d}r\Bigr)\,\mathrm{d}s\end{aligned} \tag{2.3}$$

for $t \geqslant t_0$. By the latter half of the condition (1.9), the first term of the right-hand side of (2.3) converges as $t \to \infty$. It is clear that the second term of the right-hand side of (2.3) converges as $t \to \infty$. The last term of the right-hand side of (2.3) also converges as $t \to \infty$. In fact, by integration by parts and the latter half of the condition (1.9) we find that

$$\begin{aligned}\int_{t_0}^t \mathrm{e}^{-\beta s}\Bigl(\int_{t_0}^s \mathrm{e}^{2\beta r}|\Phi_\lambda(r)|\,\mathrm{d}r\Bigr)\,\mathrm{d}s ={}& -\frac{1}{\beta}\mathrm{e}^{-\beta t}\int_{t_0}^t \mathrm{e}^{2\beta s}|\Phi_\lambda(s)|\,\mathrm{d}s\\ &+ \frac{1}{\beta}\int_{t_0}^t \mathrm{e}^{-\beta s}\mathrm{e}^{2\beta s}|\Phi_\lambda(s)|\,\mathrm{d}s\\ \leqslant{}& \frac{1}{\beta}\int_{t_0}^\infty \mathrm{e}^{\beta s}|\Phi_\lambda(s)|\,\mathrm{d}s < \infty.\end{aligned}$$

Therefore we get

$$\int_{t_0}^\infty \mathrm{e}^{-\beta s}|\Psi_\lambda(s)|\,\mathrm{d}s < \infty.$$

This proves that (1.9) implies (1.11).

Conversely, suppose that (1.11) holds. Integration by parts gives

$$\begin{aligned}(2.4)\qquad \int_{t_0}^{t} \mathrm{e}^{-\beta s} b(s)\,\mathrm{d}s &= \int_{t_0}^{t} \mathrm{e}^{-2\beta s}\mathrm{e}^{\beta s} b(s)\,\mathrm{d}s = \mathrm{e}^{-2\beta t}\Psi_\lambda(t) + 2\beta\int_{t_0}^{t} \mathrm{e}^{-2\beta s}\Psi_\lambda(s)\,\mathrm{d}s \\ &= \mathrm{e}^{-\beta t}(\mathrm{e}^{-\beta t}\Psi_\lambda(t)) + 2\beta\int_{t_0}^{t} \mathrm{e}^{-\beta s}(\mathrm{e}^{-\beta s}\Psi_\lambda(s))\,\mathrm{d}s\end{aligned}$$

for $t \geqslant t_0$. By the former half of the condition (1.11), the first term of the last member of (2.4) tends to 0 and the second term is convergent as $t \to \infty$. Therefore, (1.8) holds. Then, as in the above calculation, we find that

$$\Phi_\lambda(t) = \int_t^\infty \mathrm{e}^{-\beta s} b(s)\,\mathrm{d}s = 2\beta\int_t^\infty \mathrm{e}^{-2\beta s}\left(\int_t^s \mathrm{e}^{\beta r} b(r)\,\mathrm{d}r\right)\mathrm{d}s, \quad t \geqslant t_0,$$

and hence

$$\Phi_\lambda(t) = -\mathrm{e}^{-2\beta t}\Psi_\lambda(t) + 2\beta\int_t^\infty \mathrm{e}^{-2\beta s}\Psi_\lambda(s)\,\mathrm{d}s, \quad t \geqslant t_0.$$

Thus we have

$$(2.5)\qquad \mathrm{e}^{\beta t}|\Phi_\lambda(t)| \leqslant \mathrm{e}^{-\beta t}|\Psi_\lambda(t)| + \frac{2\beta}{\mathrm{e}^{-\beta t}}\int_t^\infty \mathrm{e}^{-2\beta s}|\Psi_\lambda(s)|\,\mathrm{d}s, \quad t \geqslant t_0.$$

By the former half of (1.11), the first term of the right-hand side of (2.5) tends to 0 as $t \to \infty$. The last term of the right-hand side of (2.5) also tends to 0 ($t \to \infty$). In fact, by l'Hospital's rule we have

$$\begin{aligned}(2.6)\qquad \lim_{t\to\infty}\frac{1}{\mathrm{e}^{-\beta t}}\int_t^\infty \mathrm{e}^{-2\beta s}|\Psi_\lambda(s)|\,\mathrm{d}s &= \lim_{t\to\infty}\frac{1}{-\beta\mathrm{e}^{-\beta t}}(-\mathrm{e}^{-2\beta t}|\Psi_\lambda(t)|) \\ &= \frac{1}{\beta}\lim_{t\to\infty}\mathrm{e}^{-\beta t}|\Psi_\lambda(t)| = 0.\end{aligned}$$

Therefore we get $\mathrm{e}^{\beta t}\Phi_\lambda(t) \to 0$ as $t \to \infty$.

From (2.5) it follows that

$$(2.7)\qquad \int_{t_0}^{t} \mathrm{e}^{\beta s}|\Phi_\lambda(s)|\,\mathrm{d}s \leqslant \int_{t_0}^{t} \mathrm{e}^{-\beta s}|\Psi_\lambda(s)|\,\mathrm{d}s + 2\beta\int_{t_0}^{t} \mathrm{e}^{\beta s}\left(\int_s^\infty \mathrm{e}^{-2\beta r}|\Psi_\lambda(r)|\,\mathrm{d}r\right)\mathrm{d}s$$

for $t \geqslant t_0$. By the latter half of (1.11), the first term of the right-hand side of (2.7) converges as $t \to \infty$. The second term of the right-hand side of (2.7) also converges as $t \to \infty$. Indeed, by integration by parts, it is seen that

$$
\begin{aligned}
\text{(2.8)} \qquad \int_{t_0}^{t} \mathrm{e}^{\beta s}\left(\int_s^\infty \mathrm{e}^{-2\beta r}|\Psi_\lambda(r)|\,\mathrm{d}r\right)\mathrm{d}s = {} & \frac{1}{\beta \mathrm{e}^{-\beta t}} \int_t^\infty \mathrm{e}^{-2\beta s}|\Psi_\lambda(s)|\,\mathrm{d}s \\
& - \frac{1}{\beta \mathrm{e}^{-\beta t_0}} \int_{t_0}^\infty \mathrm{e}^{-2\beta s}|\Psi_\lambda(s)|\,\mathrm{d}s \\
& + \frac{1}{\beta} \int_{t_0}^{t} \mathrm{e}^{-\beta s}|\Psi_\lambda(s)|\,\mathrm{d}s.
\end{aligned}
$$

Since we have (2.6), the first term of the right-hand side of (2.8) tends to 0 as $t \to \infty$. From the latter half of (1.11), the last term of the right-hand side of (2.8) converges as $t \to \infty$. Consequently we obtain

$$\int_{t_0}^\infty \mathrm{e}^{\beta s}|\Phi_\lambda(s)|\,\mathrm{d}s < \infty.$$

This shows that (1.11) implies (1.9). The proof of Lemma 2.1 is complete. $\square$

Lemma 2.2.

(I) *Suppose that* (1.9) *holds. Define the function* $\widehat{\Phi}_\lambda(t)$ *by*

$$\text{(2.9)} \qquad \widehat{\Phi}_\lambda(t) = \int_t^\infty \mathrm{e}^{(\alpha+1)\lambda s}|\Phi_\lambda(s)|^2\,\mathrm{d}s, \quad t \geqslant t_0.$$

Then we have

$$\text{(2.10)} \qquad \lim_{t\to\infty} \mathrm{e}^{(\alpha+1)\lambda t}\widehat{\Phi}_\lambda(t) = 0$$

and

$$\text{(2.11)} \qquad \int_{t_0}^\infty \mathrm{e}^{(\alpha+1)\lambda s}\widehat{\Phi}_\lambda(s)\,\mathrm{d}s < \infty.$$

(II) *Suppose that* (1.11) *holds. Define the function* $\widehat{\Psi}_\lambda(t)$ *by*

$$\text{(2.12)} \qquad \widehat{\Psi}_\lambda(t) = \int_{t_0}^t \mathrm{e}^{-(\alpha+1)\lambda s}|\Psi_\lambda(s)|^2\,\mathrm{d}s, \quad t \geqslant t_0.$$

Then we have

$$\text{(2.13)} \qquad \lim_{t\to\infty} \mathrm{e}^{-(\alpha+1)\lambda t}\widehat{\Psi}_\lambda(t) = 0$$

and

$$\text{(2.14)} \qquad \int_{t_0}^\infty \mathrm{e}^{-(\alpha+1)\lambda s}\widehat{\Psi}_\lambda(s)\,\mathrm{d}s < \infty.$$

P r o o f. Let β be the constant defined by (2.1).

(I) Suppose that (1.9) holds. The function $\widehat{\Phi}_\lambda(t)$ is well defined since $\mathrm{e}^{\beta t}|\Phi_\lambda(t)|$ is integrable on $[t_0,\infty)$ (the latter half of (1.9)) and $\Phi_\lambda(t)\to 0$ as $t\to\infty$. By l'Hospital's rule and the former half of (1.9), we see that

$$\lim_{t\to\infty}\mathrm{e}^{\beta t}\widehat{\Phi}_\lambda(t)=\lim_{t\to\infty}\frac{1}{\mathrm{e}^{-\beta t}}\int_t^\infty \mathrm{e}^{\beta s}|\Phi_\lambda(s)|^2\,\mathrm{d}s=\frac{1}{\beta}\lim_{t\to\infty}(\mathrm{e}^{\beta t}|\Phi_\lambda(t)|)^2=0,$$

which proves (2.10). Integration by parts gives

$$\int_{t_0}^t \mathrm{e}^{\beta s}\widehat{\Phi}_\lambda(s)\,\mathrm{d}s=\frac{1}{\beta}\mathrm{e}^{\beta t}\widehat{\Phi}_\lambda(t)-\frac{1}{\beta}\mathrm{e}^{\beta t_0}\widehat{\Phi}_\lambda(t_0)+\frac{1}{\beta}\int_{t_0}^t \mathrm{e}^{2\beta s}|\Phi_\lambda(s)|^2\,\mathrm{d}s$$

for $t\geqslant t_0$. The first term of the right-hand side of the above equality tends to 0 as $t\to\infty$ by (2.10). Since $\mathrm{e}^{\beta t}|\Phi_\lambda(t)|\to 0$ $(t\to\infty)$, we have

$$\mathrm{e}^{2\beta t}|\Phi_\lambda(t)|^2\leqslant \mathrm{e}^{\beta t}|\Phi_\lambda(t)|$$

for all large t. Since $\mathrm{e}^{\beta t}|\Phi_\lambda(t)|$ is integrable on $[t_0,\infty)$, the function $\mathrm{e}^{2\beta t}|\Phi_\lambda(t)|^2$ is also integrable on $[t_0,\infty)$. Consequently, (2.11) holds.

(II) Suppose that (1.11) holds. By l'Hospital's rule and the former half of (1.11) we see that

$$\lim_{t\to\infty}\mathrm{e}^{-\beta t}\widehat{\Psi}_\lambda(t)=\lim_{t\to\infty}\frac{1}{\mathrm{e}^{\beta t}}\int_{t_0}^t \mathrm{e}^{-\beta s}|\Psi_\lambda(s)|^2\,\mathrm{d}s=\frac{1}{\beta}\lim_{t\to\infty}(\mathrm{e}^{-\beta t}|\Psi_\lambda(t)|)^2=0,$$

proving (2.13). By using integration by parts we have

$$\begin{aligned}\int_{t_0}^t \mathrm{e}^{-\beta s}\widehat{\Psi}_\lambda(s)\,\mathrm{d}s&=-\frac{1}{\beta}\mathrm{e}^{-\beta t}\widehat{\Psi}_\lambda(t)+\frac{1}{\beta}\int_{t_0}^t \mathrm{e}^{-2\beta s}|\Psi_\lambda(s)|^2\,\mathrm{d}s\\&\leqslant\frac{1}{\beta}\int_{t_0}^t \mathrm{e}^{-2\beta s}|\Psi_\lambda(s)|^2\,\mathrm{d}s,\quad t\geqslant t_0.\end{aligned}$$

Since $\mathrm{e}^{-\beta t}|\Psi_\lambda(t)|\to 0$ as $t\to\infty$, we have

$$\mathrm{e}^{-2\beta t}|\Psi_\lambda(t)|^2\leqslant \mathrm{e}^{-\beta t}|\Psi_\lambda(t)|$$

for all large t. Since $\mathrm{e}^{-\beta t}|\Psi_\lambda(t)|$ is integrable on $[t_0,\infty)$ (the latter half of (1.11)), the function $\mathrm{e}^{-2\beta t}|\Psi_\lambda(t)|^2$ is integrable on $[t_0,\infty)$. Consequently, (2.14) holds. The proof of Lemma 2.2 is complete. $\square$

We use the asterisk notation

$$\xi^{\alpha *} = |\xi|^{\alpha} \operatorname{sgn} \xi, \quad \xi \in \mathbb{R},\ \alpha > 0.$$

It is easy to see that, for $\xi, \eta, \xi_1, \xi_2 \in \mathbb{R}$ and $\alpha, \alpha_1, \alpha_2 > 0$,

- ▷ $(\xi\eta)^{\alpha *} = \xi^{\alpha *}\eta^{\alpha *}$, $(-\xi)^{\alpha *} = -\xi^{\alpha *}$;
- ▷ $(\xi^{\alpha_1 *})^{\alpha_2 *} = \xi^{(\alpha_1\alpha_2)*}$, $(\xi^{\alpha *})^{(1/\alpha)*} = \xi$, $(\xi^{(1/\alpha)*})^{\alpha *} = \xi$;
- ▷ $\xi^{\alpha *} = \eta$ if and only if $\xi = \eta^{(1/\alpha)*}$;
- ▷ $\xi_1^{\alpha *} \leqslant \xi_2^{\alpha *}$ if and only if $\xi_1 \leqslant \xi_2$; $\xi_1^{\alpha *} < \xi_2^{\alpha *}$ if and only if $\xi_1 < \xi_2$;
- ▷ $f(\xi) = \xi^{\alpha *}$ is a continuous function of $\xi \in \mathbb{R}$.

With this asterisk notation, the equation (1.1) is rewritten in the form

$$(u'^{\alpha *})' = \alpha(\lambda^{\alpha+1} + b(t))u^{\alpha *}, \quad t \geqslant t_0. \tag{3.1}$$

Throughout the paper the following fact plays an essential part. Let $u(t)$ be a nonoscillatory solution of (3.1). We may suppose that $u(t) > 0$ for $t \geqslant T$ $(> t_0)$. Put

$$v(t) = \left(\frac{u'(t)}{u(t)}\right)^{\alpha *}, \quad t \geqslant T. \tag{3.2}$$

Then $v(t)$ satisfies the generalized Riccati differential equation

$$v'(t) = \alpha(\lambda^{\alpha+1} + b(t)) - \alpha|v(t)|^{(\alpha+1)/\alpha}, \quad t \geqslant T. \tag{3.3}$$

Conversely, if $v(t)$ is a solution of (3.3) on $[T, \infty)$, then

$$u(t) = \exp\left(\int_T^t v(s)^{(1/\alpha)*}\, \mathrm{d}s\right), \quad t \geqslant T, \tag{3.4}$$

is a positive solution of (3.1) on $[T, \infty)$. The proof is immediate.

If a nonoscillatory solution $u(t)$ of (3.1) on $[T, \infty)$ satisfies the asymptotic condition of the type (1.4), then the function $v(t)$ which is defined by (3.2) satisfies $\lim v(t) = -\lambda^{\alpha}$ as $t \to \infty$. Put $w(t) = -\lambda^{\alpha} - v(t)$ for $t \geqslant T$. From (3.3) it is evident that

$$w'(t) = -\alpha(\lambda^{\alpha+1} + b(t)) + \alpha|\lambda^{\alpha} + w(t)|^{(\alpha+1)/\alpha}, \quad t \geqslant T. \tag{3.5}$$

It is also clear that $\lim\limits_{t\to\infty} w(t) = 0$. Therefore we can suppose that $|w(t)| \leqslant \frac{1}{2}\lambda^{\alpha}$ for $t \geqslant T$.

Define the constant β by (2.1), and the function $\varphi(w)$ by

$$\varphi(w) = (\lambda^\alpha + w)^{(\alpha+1)/\alpha} - \lambda^{\alpha+1} - \frac{\alpha+1}{\alpha}\lambda w, \quad |w| \leqslant \frac{\lambda^\alpha}{2}. \tag{3.6}$$

Note that $|\lambda^\alpha + w| = \lambda^\alpha + w$ for $|w| \leqslant \frac{1}{2}\lambda^\alpha$. Then (3.5) is rewritten as

$$w'(t) = -\alpha b(t) + \beta w(t) + \alpha\varphi(w(t)), \quad t \geqslant T. \tag{3.7}$$

Similarly, if a nonoscillatory solution $u(t)$ of (3.1) on $[T, \infty)$ satisfies the asymptotic condition of the type (1.5), then the function $v(t)$ which is defined by (3.2) satisfies $\lim v(t) = \lambda^\alpha$ as $t \to \infty$, $w(t) = -\lambda^\alpha + v(t)$ $(t \geqslant T)$ satisfies

$$w'(t) = \alpha(\lambda^{\alpha+1} + b(t)) - \alpha|\lambda^\alpha + w(t)|^{(\alpha+1)/\alpha}, \quad t \geqslant T, \tag{3.8}$$

and $\lim w(t) = 0$ as $t \to \infty$. We suppose that $|w(t)| \leqslant \frac{1}{2}\lambda^\alpha$ for $t \geqslant T$. By using the constant β given by (2.1) and the function $\varphi(w)$ given by (3.6), the equality (3.8) is rewritten in the form

$$w'(t) = \alpha b(t) - \beta w(t) - \alpha\varphi(w(t)), \quad t \geqslant T. \tag{3.9}$$

In this paper the equations (3.7) and (3.9) play an important role. For the proof of the existence of a solution $u_0(t)$ (or $u_1(t)$) of (1.1) satisfying (1.4) (or (1.5)), we use (3.7) (or (3.9)). We need the following lemmas.

Lemma 3.1. *Let $\alpha > 0$ and $\lambda > 0$ be constants. Define the function $\varphi(w)$ by* (3.6). *Then we have*

$$0 \leqslant \varphi(w) \leqslant L(\alpha,\lambda)w^2 \quad \text{and} \quad |\varphi'(w)| \leqslant 2L(\alpha,\lambda)|w|$$

for $|w| \leqslant \frac{1}{2}\lambda^\alpha$, where

$$L(\alpha,\lambda) = \begin{cases} \dfrac{\alpha+1}{2\alpha^2}\Big(\dfrac{3}{2}\Big)^{(1/\alpha)-1}\lambda^{-\alpha+1}, & 0 < \alpha \leqslant 1, \\ \dfrac{\alpha+1}{2\alpha^2}\Big(\dfrac{1}{2}\Big)^{(1/\alpha)-1}\lambda^{-\alpha+1}, & \alpha > 1. \end{cases}$$

Define the function $\psi(w)$ by

$$\psi(w) = (\lambda^\alpha + w)^{1/\alpha} - \lambda - \frac{1}{\alpha}\lambda^{-\alpha+1}w, \quad |w| \leqslant \frac{\lambda^\alpha}{2}. \tag{3.10}$$

Then we obtain the the following estimate for $\psi(w)$.

Lemma 3.2. *Let $\alpha > 0$ and $\lambda > 0$ be constants. Define the function $\psi(w)$ by* (3.10). *Then we have*

$$|\psi(w)| \leqslant M(\alpha,\lambda)w^2, \quad |w| \leqslant \frac{\lambda^\alpha}{2},$$

where

$$M(\alpha,\lambda) = \begin{cases} \dfrac{|\alpha-1|}{2\alpha^2}\Big(\dfrac{3}{2}\Big)^{(1/\alpha)-2}\lambda^{-2\alpha+1}, & 0<\alpha\leqslant \frac12, \\ \dfrac{|\alpha-1|}{2\alpha^2}\Big(\dfrac{1}{2}\Big)^{(1/\alpha)-2}\lambda^{-2\alpha+1}, & \alpha > \frac12. \end{cases}$$

For the proofs of Lemmas 3.1 and 3.2, see [11], Lemma 2.1 and Lemma 2.2.

4. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. For the proof of the existence of a solution $u_0(t)$ (or $u_1(t)$) of (1.1) satisfying (1.4) (or (1.5)) we use (1.9) (or (1.11)). As before, $\beta > 0$ is the constant given by (2.1).

Proof of Theorem 1.1. We will first prove the existence of a nonoscillatory solution $u_0(t)$ of (1.1) which satisfies (1.4). As mentioned above, we use (1.9) and solve the generalized Riccati differential equation (3.3). To this end, we utilize the equation (3.7). As an integral form of (3.7) it is natural to consider

$$w(t) = \alpha e^{\beta t}\int_t^\infty e^{-\beta s}b(s)\,ds - \alpha e^{\beta t}\int_t^\infty e^{-\beta s}\varphi(w(s))\,ds, \quad t \geqslant T, \tag{4.1}$$

where T is a suitable number. Note that the first term of the right-hand side of (4.1) is equal to $\alpha e^{\beta t}\Phi_\lambda(t)$, where $\Phi_\lambda(t)$ is given by (1.10). We define $\widehat{\Phi}_\lambda(t)$ by (2.9). By (I) of Lemma 2.2 we have (2.10) and (2.11).

By (1.9), (2.10) and (2.11), the functions $e^{\beta t}|\Phi_\lambda(t)|$ and $e^{\beta t}\widehat{\Phi}_\lambda(t)$ tend to 0 as $t \to \infty$ and are integrable on $[t_0,\infty)$. Let $L(\alpha,\lambda)$ be a positive constant appearing in Lemma 3.1. For simplicity of notation, we put

$$\gamma = 3\alpha^3 L(\alpha,\lambda). \tag{4.2}$$

It is possible to take a number $T > t_0$ so that

$$\alpha e^{\beta t}|\Phi_\lambda(t)| + \gamma e^{\beta t}\widehat{\Phi}_\lambda(t) \leqslant \frac{\lambda^\alpha}{2}, \quad t \geqslant T, \tag{4.3}$$

and

$$\alpha L(\alpha,\lambda)\int_T^\infty (\alpha e^{\beta s}|\Phi_\lambda(s)| + \gamma e^{\beta s}\widehat{\Phi}_\lambda(s))\,ds \leqslant \frac16. \tag{4.4}$$

Then, by (4.4), we have

$$\alpha L(\alpha,\lambda)\gamma\int_T^\infty e^{\beta s}\widehat{\Phi}_\lambda(s)\,ds \leqslant \frac{1}{6}. \tag{4.5}$$

Let $C_B[T,\infty)$ denote the Banach space of all bounded continuous functions $w(t)$ on $[T,\infty)$ with the supremum norm

$$\|w\| = \sup_{t\geqslant T}|w(t)|. \tag{4.6}$$

Define the set W by

$$W = \{w \in C_B[T,\infty)\colon\ |w(t)| \leqslant \alpha e^{\beta t}|\Phi_\lambda(t)| + \gamma e^{\beta t}\widehat{\Phi}_\lambda(t) \text{ for } t \geqslant T\}.$$

If $w \in W$, then $|w(t)| \leqslant \frac{1}{2}\lambda^\alpha$ for $t \geqslant T$ (see (4.3)). The set W is a nonempty closed subset of $C_B[T,\infty)$. Define the integral operator $\mathcal{F}$ on W by

$$(\mathcal{F}w)(t) = \alpha e^{\beta t}\int_t^\infty e^{-\beta s}b(s)\,ds - \alpha e^{\beta t}\int_t^\infty e^{-\beta s}\varphi(w(s))\,ds, \quad t \geqslant T,$$

where $\varphi(w)$ is given by (3.6). Observe that $\mathcal{F}w$ is well-defined for $w \in W$ and that $(\mathcal{F}w)(t)$ is a continuous function on $[T,\infty)$.

Let $w \in W$. By Lemma 3.1 and the definition of $\mathcal{F}w$, we get

$$\begin{aligned}|(\mathcal{F}w)(t)| &\leqslant \alpha e^{\beta t}|\Phi_\lambda(t)| + \alpha e^{\beta t}\int_t^\infty e^{-\beta s}|\varphi(w(s))|\,ds\\ &\leqslant \alpha e^{\beta t}|\Phi_\lambda(t)| + \alpha L(\alpha,\lambda)e^{\beta t}\int_t^\infty e^{-\beta s}|w(s)|^2\,ds\end{aligned}$$

for $t \geqslant T$. As a general inequality we have $(A+B)^2 \leqslant 2A^2 + 2B^2$ for all real numbers A and B. Therefore, if $w \in W$, then

$$\begin{aligned}\int_t^\infty e^{-\beta s}|w(s)|^2\,ds &\leqslant \int_t^\infty e^{-\beta s}(2(\alpha e^{\beta s}|\Phi_\lambda(s)|)^2 + 2(\gamma e^{\beta s}\widehat{\Phi}_\lambda(s))^2)\,ds\\ &\leqslant 2\alpha^2\widehat{\Phi}_\lambda(t) + 2\gamma^2\widehat{\Phi}_\lambda(t)\int_t^\infty e^{\beta s}\widehat{\Phi}_\lambda(s)\,ds\end{aligned}$$

for $t \geqslant T$. Here, in the last step, the decreasing property of $\widehat{\Phi}_\lambda(t)$ has been used. Hence, in view of (4.2) and (4.5), we obtain

$$\begin{aligned}|(\mathcal{F}w)(t)| &\leqslant \alpha e^{\beta t}|\Phi_\lambda(t)| + \frac{2}{3}\gamma e^{\beta t}\widehat{\Phi}_\lambda(t) + 2\alpha L(\alpha,\lambda)\gamma^2 e^{\beta t}\widehat{\Phi}_\lambda(t)\int_t^\infty e^{\beta s}\widehat{\Phi}_\lambda(s)\,ds\\ &\leqslant \alpha e^{\beta t}|\Phi_\lambda(t)| + \gamma e^{\beta t}\widehat{\Phi}_\lambda(t), \quad t \geqslant T.\end{aligned}$$

This implies that $\mathcal{F}$ maps W into W.

Furthermore, it can be proved that $\mathcal{F}$ is a contraction mapping on W. In fact, if $w_1, w_2 \in W$, then

$$|(\mathcal{F}w_1)(t) - (\mathcal{F}w_2)(t)| \leqslant \alpha \mathrm{e}^{\beta t} \int_t^\infty \mathrm{e}^{-\beta s} |\varphi(w_1(s)) - \varphi(w_2(s))| \, \mathrm{d}s, \quad t \geqslant T.$$

The mean value theorem implies that there exists $\theta(t) \in (0,1)$ such that

$$\varphi(w_1(t)) - \varphi(w_2(t)) = \varphi'(\xi(t))(w_1(t) - w_2(t))$$

with

$$\xi(t) = w_1(t) + \theta(t)(w_2(t) - w_1(t)).$$

Then, noting that

$$|\xi(t)| \leqslant \alpha \mathrm{e}^{\beta t} |\Phi_\lambda(t)| + \gamma \mathrm{e}^{\beta t} \widehat{\Phi}_\lambda(t), \quad t \geqslant T,$$

and using the estimate for $|\varphi'(w)|$ in Lemma 3.1, we see that

$$|\varphi(w_1(t)) - \varphi(w_2(t))| \leqslant 2L(\alpha, \lambda)(\alpha \mathrm{e}^{\beta t} |\Phi_\lambda(t)| + \gamma \mathrm{e}^{\beta t} \widehat{\Phi}_\lambda(t)) |w_1(t) - w_2(t)|$$

for $t \geqslant T$. Therefore we find that

$$\begin{aligned}
&|(\mathcal{F}w_1)(t) - (\mathcal{F}w_2)(t)| \\
&\quad \leqslant 2\alpha L(\alpha, \lambda) \mathrm{e}^{\beta t} \int_t^\infty \mathrm{e}^{-\beta s} (\alpha \mathrm{e}^{\beta s} |\Phi_\lambda(s)| + \gamma \mathrm{e}^{\beta s} \widehat{\Phi}_\lambda(s)) \, \mathrm{d}s \|w_1 - w_2\| \\
&\quad \leqslant 2\alpha L(\alpha, \lambda) \int_t^\infty (\alpha \mathrm{e}^{\beta s} |\Phi_\lambda(s)| + \gamma \mathrm{e}^{\beta s} \widehat{\Phi}_\lambda(s)) \, \mathrm{d}s \|w_1 - w_2\|
\end{aligned}$$

for $t \geqslant T$. Therefore it follows from (4.4) that

$$|(\mathcal{F}w_1)(t) - (\mathcal{F}w_2)(t)| \leqslant \frac{1}{3} \|w_1 - w_2\|, \quad t \geqslant T,$$

which yields

$$\|\mathcal{F}w_1 - \mathcal{F}w_2\| \leqslant \frac{1}{3} \|w_1 - w_2\|.$$

Thus, $\mathcal{F}$ is a contraction mapping on W as claimed.

By the contraction mapping principle, $\mathcal{F}$ has a fixed element $w \in W$. This fixed element $w(t)$ satisfies (4.1), and so it satisfies (3.7). By the definition of $\varphi(w)$ we find that $w(t)$ satisfies (3.5). By the construction of $w(t)$ we have

$$|w(t)| \leqslant \alpha \mathrm{e}^{\beta t} |\Phi_\lambda(t)| + \gamma \mathrm{e}^{\beta t} \widehat{\Phi}_\lambda(t) \leqslant \frac{\lambda^\alpha}{2}, \quad t \geqslant T, \tag{4.7}$$

which gives $\lambda^\alpha + w(t) \geqslant \frac{1}{2}\lambda^\alpha > 0$ for $t \geqslant T$. The function $v(t)$ defined by $v(t) = -\lambda^\alpha - w(t)$ $(t \geqslant T)$ satisfies (3.3) and $v(t) \leqslant -\frac{1}{2}\lambda^\alpha < 0$ for $t \geqslant T$. For this $v(t)$, define $u(t)$ by (3.4). It is seen that $u(t)$ is a positive solution of (3.1) on $[T, \infty)$. Using the function $\psi(w)$ given by (3.10), we have

$$\frac{u'(t)}{u(t)} = v(t)^{(1/\alpha)*} = -(\lambda^\alpha + w(t))^{1/\alpha} = -\lambda - \frac{\lambda^{-\alpha+1}}{\alpha} w(t) - \psi(w(t)) \tag{4.8}$$

for $t \geqslant T$ and integration of (4.8) gives

$$\log \frac{u(t)}{u(T)} = -\lambda(t - T) - \frac{\lambda^{-\alpha+1}}{\alpha} \int_T^t w(s)\,\mathrm{d}s - \int_T^t \psi(w(s))\,\mathrm{d}s \tag{4.9}$$

for $t \geqslant T$. Since $\mathrm{e}^{\beta t}|\Phi_\lambda(t)|$ and $\mathrm{e}^{\beta t}\widehat{\Phi}_\lambda(t)$ are integrable on $[t_0, \infty)$, the inequality (4.7) implies

$$\int_T^\infty |w(s)|\,\mathrm{d}s < \infty. \tag{4.10}$$

From Lemma 3.2 and (4.7) it is seen that

$$|\psi(w(t))| \leqslant M(\alpha, \lambda) w(t)^2, \quad t \geqslant T.$$

Since $w(t) \to 0$ as $t \to \infty$ (see (4.7)), we have $w(t)^2 \leqslant |w(t)|$ for all large t. Therefore, (4.10) gives

$$\int_T^\infty |\psi(w(s))|\,\mathrm{d}s < \infty. \tag{4.11}$$

By (4.9), the solution $u(t)$ is written in the form

$$u(t) = c(t)\mathrm{e}^{-\lambda t}, \quad t \geqslant T,$$

where

$$c(t) = u(T) \exp\biggl(\lambda T - \frac{\lambda^{-\alpha+1}}{\alpha} \int_T^t w(s)\,\mathrm{d}s - \int_T^t \psi(w(s))\,\mathrm{d}s\biggr), \quad t \geqslant T.$$

Then, from (4.10) and (4.11), it is clear that $c(t)$ has a positive finite limit as $t \to \infty$. Put $\lim_{t\to\infty} c(t) = c_0$ (> 0). Since $\lim_{t\to\infty} w(t) = 0$, the equality (4.8) gives

$$\lim_{t\to\infty} \frac{u'(t)}{u(t)} = -\lambda.$$

Since $u(t)/\mathrm{e}^{-\lambda t} = c(t) \to c_0$ $(t \to \infty)$, the above equality implies $u'(t)/\mathrm{e}^{-\lambda t} \to -\lambda c_0$ $(t \to \infty)$. Then the function $u_0(t) = u(t)/c_0$ is a solution of (1.1) and satisfies (1.4).

Next we prove the existence of a nonoscillatory solution $u_1(t)$ which satisfies (1.5). For this purpose, we use (1.11) and solve the equation (3.9). An integral form of (3.9) is

$$w(t) = \mathrm{e}^{-\beta(t-T)} w(T) + \alpha \mathrm{e}^{-\beta t} \int_T^t \mathrm{e}^{\beta s} b(s)\,\mathrm{d}s - \alpha \mathrm{e}^{-\beta t} \int_T^t \mathrm{e}^{\beta s} \varphi(w(s))\,\mathrm{d}s$$

for $t \geqslant T$, where T is a suitable number. If we take $w(T) = \alpha \mathrm{e}^{-\beta T} \Psi_\lambda(T)$, where $\Psi_\lambda(t)$ is defined by (1.12), then the above equality becomes

$$w(t) = \alpha \mathrm{e}^{-\beta t} \int_{t_0}^t \mathrm{e}^{\beta s} b(s)\,\mathrm{d}s - \alpha \mathrm{e}^{-\beta t} \int_T^t \mathrm{e}^{\beta s} \varphi(w(s))\,\mathrm{d}s, \quad t \geqslant T. \tag{4.12}$$

The first term of the right-hand side of (4.12) is equal to $\alpha \mathrm{e}^{-\beta t} \Psi_\lambda(t)$.

Now, define $\widehat{\Psi}_\lambda(t)$ by (2.12). By (II) of Lemma 2.2, we have (2.13) and (2.14). From (1.11), (2.13) and (2.14), the functions $\mathrm{e}^{-\beta t}|\Psi_\lambda(t)|$ and $\mathrm{e}^{-\beta t}\widehat{\Psi}_\lambda(t)$ tend to 0 as $t \to \infty$ and are integrable on $[t_0, \infty)$. Therefore it is possible to choose a number $T > t_0$ so that

$$\alpha \mathrm{e}^{-\beta t} |\Psi_\lambda(t)| + \gamma \mathrm{e}^{-\beta t} \widehat{\Psi}_\lambda(t) \leqslant \frac{\lambda^\alpha}{2}, \quad t \geqslant T, \tag{4.13}$$

and

$$\alpha L(\alpha, \lambda) \int_T^\infty (\alpha \mathrm{e}^{-\beta s} |\Psi_\lambda(s)| + \gamma \mathrm{e}^{-\beta s} \widehat{\Psi}_\lambda(s))\,\mathrm{d}s \leqslant \frac{1}{6}, \tag{4.14}$$

where γ is given by (4.2) and $L(\alpha, \lambda)$ is a positive constant appearing in Lemma 3.1. The inequality (4.14) implies

$$\alpha L(\alpha, \lambda) \gamma \int_T^\infty \mathrm{e}^{-\beta s} \widehat{\Psi}_\lambda(s)\,\mathrm{d}s \leqslant \frac{1}{6}. \tag{4.15}$$

Let $C_B[T, \infty)$ denote the Banach space of all bounded continuous functions $w(t)$ on $[T, \infty)$ with the supremum norm (4.6). Define the set W by

$$W = \{w \in C_B[T, \infty) \colon\ |w(t)| \leqslant \alpha \mathrm{e}^{-\beta t} |\Psi_\lambda(t)| + \gamma \mathrm{e}^{-\beta t} \widehat{\Psi}_\lambda(t) \text{ for } t \geqslant T\}.$$

If $w \in W$, then $|w(t)| \leqslant \frac{1}{2}\lambda^\alpha$ for $t \geqslant T$ (see (4.13)). The set W is a nonempty closed subset of $C_B[T, \infty)$. Define the operator $\mathcal{F}$ on W by

$$(\mathcal{F}w)(t) = \alpha \mathrm{e}^{-\beta t} \int_{t_0}^t \mathrm{e}^{\beta s} b(s)\,\mathrm{d}s - \alpha \mathrm{e}^{-\beta t} \int_T^t \mathrm{e}^{\beta s} \varphi(w(s))\,\mathrm{d}s, \quad t \geqslant T,$$

where $\varphi(w)$ is given by (3.6). It is clear that $(\mathcal{F}w)(t)$ is a continuous function on $[T, \infty)$.

Let $w \in W$. By Lemma 3.1,

$$
\begin{aligned}
|(\mathcal{F}w)(t)| &\leqslant \alpha \mathrm{e}^{-\beta t}|\Psi_\lambda(t)| + \alpha \mathrm{e}^{-\beta t}\int_T^t \mathrm{e}^{\beta s}|\varphi(w(s))|\,\mathrm{d}s \\
&\leqslant \alpha \mathrm{e}^{-\beta t}|\Psi_\lambda(t)| + \alpha L(\alpha,\lambda)\mathrm{e}^{-\beta t}\int_T^t \mathrm{e}^{\beta s}|w(s)|^2\,\mathrm{d}s
\end{aligned}
$$

for $t \geqslant T$. It is found that

$$
\begin{aligned}
\int_T^t \mathrm{e}^{\beta s}|w(s)|^2\,\mathrm{d}s &\leqslant \int_T^t \mathrm{e}^{\beta s}(2(\alpha \mathrm{e}^{-\beta s}|\Psi_\lambda(s)|)^2 + 2(\gamma \mathrm{e}^{-\beta s}\widehat{\Psi}_\lambda(s))^2)\,\mathrm{d}s \\
&\leqslant 2\alpha^2\widehat{\Psi}_\lambda(t) + 2\gamma^2\widehat{\Psi}_\lambda(t)\int_T^t \mathrm{e}^{-\beta s}\widehat{\Psi}_\lambda(s)\,\mathrm{d}s
\end{aligned}
$$

for $t \geqslant T$. Here, in the last step, the increasing property of $\widehat{\Psi}_\lambda(t)$ has been used. Hence, by (4.2) and (4.15), we obtain

$$
\begin{aligned}
|(\mathcal{F}w)(t)| &\leqslant \alpha \mathrm{e}^{-\beta t}|\Psi_\lambda(t)| + \frac{2}{3}\gamma \mathrm{e}^{-\beta t}\widehat{\Psi}_\lambda(t) + 2\alpha L(\alpha,\lambda)\gamma^2 \mathrm{e}^{-\beta t}\widehat{\Psi}_\lambda(t)\int_T^t \mathrm{e}^{-\beta s}\widehat{\Psi}_\lambda(s)\,\mathrm{d}s \\
&\leqslant \alpha \mathrm{e}^{-\beta t}|\Psi_\lambda(t)| + \gamma \mathrm{e}^{-\beta t}\widehat{\Psi}_\lambda(t), \quad t \geqslant T.
\end{aligned}
$$

This implies that $\mathcal{F}$ maps W into itself. Moreover, it is shown that $\mathcal{F}$ is a contraction mapping on W. In fact, if $w_1, w_2 \in W$, then

$$
|(\mathcal{F}w_1)(t) - (\mathcal{F}w_2)(t)| \leqslant \alpha \mathrm{e}^{-\beta t}\int_T^t \mathrm{e}^{\beta s}|\varphi(w_1(s)) - \varphi(w_2(s))|\,\mathrm{d}s, \quad t \geqslant T.
$$

As in the previous calculation we have

$$
|\varphi(w_1(t)) - \varphi(w_2(t))| \leqslant 2L(\alpha,\lambda)(\alpha \mathrm{e}^{-\beta t}|\Psi_\lambda(t)| + \gamma \mathrm{e}^{-\beta t}\widehat{\Psi}_\lambda(t))|w_1(t) - w_2(t)|
$$

for $t \geqslant T$. Therefore we find that

$$
\begin{aligned}
&|(\mathcal{F}w_1)(t) - (\mathcal{F}w_2)(t)| \\
&\qquad \leqslant 2\alpha L(\alpha,\lambda)\mathrm{e}^{-\beta t}\int_T^t \mathrm{e}^{\beta s}(\alpha \mathrm{e}^{-\beta s}|\Psi_\lambda(s)| + \gamma \mathrm{e}^{-\beta s}\widehat{\Psi}_\lambda(s))\,\mathrm{d}s\|w_1 - w_2\| \\
&\qquad \leqslant 2\alpha L(\alpha,\lambda)\int_T^t (\alpha \mathrm{e}^{-\beta s}|\Psi_\lambda(s)| + \gamma \mathrm{e}^{-\beta s}\widehat{\Psi}_\lambda(s))\,\mathrm{d}s\|w_1 - w_2\|
\end{aligned}
$$

for $t \geqslant T$. Hence, (4.14) yields

$$
\|\mathcal{F}w_1 - \mathcal{F}w_2\| \leqslant \frac{1}{3}\|w_1 - w_2\|.
$$

Thus, $\mathcal{F}$ is a contraction mapping on W.

By the contraction mapping principle, $\mathcal{F}$ has a fixed element $w \in W$. This fixed element $w(t)$ satisfies (4.12), and so it satisfies (3.9). By the definition of $\varphi(w)$ we see that $w(t)$ satisfies (3.8). Therefore the function $v(t) = \lambda^\alpha + w(t)$ satisfies (3.3). By the construction of $w(t)$ we have

$$(4.16) \qquad |w(t)| \leqslant \alpha \mathrm{e}^{-\beta t}|\Psi_\lambda(t)| + \gamma \mathrm{e}^{-\beta t}\widehat{\Psi}_\lambda(t) \leqslant \frac{\lambda^\alpha}{2}, \quad t \geqslant T,$$

and hence $v(t) \geqslant \frac{1}{2}\lambda^\alpha > 0$ for $t \geqslant T$. For this $v(t)$, define $u(t)$ by (3.4). It is seen that $u(t)$ is a positive solution of (3.1) on $[T, \infty)$. Using the function $\psi(w)$ defined by (3.10), we have

$$(4.17) \qquad \frac{u'(t)}{u(t)} = v(t)^{(1/\alpha)*} = (\lambda^\alpha + w(t))^{1/\alpha} = \lambda + \frac{\lambda^{-\alpha+1}}{\alpha} w(t) + \psi(w(t))$$

for $t \geqslant T$ and integration of (4.17) gives

$$(4.18) \qquad \log \frac{u(t)}{u(T)} = \lambda(t - T) + \frac{\lambda^{-\alpha+1}}{\alpha} \int_T^t w(s)\,\mathrm{d}s + \int_T^t \psi(w(s))\,\mathrm{d}s$$

for $t \geqslant T$. Since $\mathrm{e}^{-\beta t}|\Psi_\lambda(t)|$ and $\mathrm{e}^{-\beta t}\widehat{\Psi}_\lambda(t)$ are integrable on $[t_0, \infty)$, the inequality (4.16) implies

$$\int_T^\infty |w(s)|\,\mathrm{d}s < \infty.$$

By Lemma 3.2 and (4.16) we have

$$|\psi(w(t))| \leqslant M(\alpha, \lambda) w(t)^2, \quad t \geqslant T.$$

Since $w(t) \to 0$ as $t \to \infty$ (see (4.16)), we have $w(t)^2 \leqslant |w(t)|$ for all large t. Therefore, the integrability of $|w(t)|$ on $[T, \infty)$ gives

$$\int_T^\infty |\psi(w(s))|\,\mathrm{d}s < \infty.$$

By (4.18), the solution $u(t)$ is written in the form

$$u(t) = c(t)\mathrm{e}^{\lambda t}, \quad t \geqslant T,$$

where

$$c(t) = u(T) \exp\biggl(-\lambda T + \frac{\lambda^{-\alpha+1}}{\alpha} \int_T^t w(s)\,\mathrm{d}s + \int_T^t \psi(w(s))\,\mathrm{d}s\biggr), \quad t \geqslant T.$$

Then it is clear that $c(t)$ has a positive finite limit as $t \to \infty$. Put $\lim_{t\to\infty} c(t) = c_1$ (>0). Since $\lim_{t\to\infty} w(t) = 0$, the equality (4.17) gives

$$\lim_{t\to\infty} \frac{u'(t)}{u(t)} = \lambda.$$

Since $u(t)/\mathrm{e}^{\lambda t} = c(t) \to c_1$ $(t \to \infty)$, the above equality implies $u'(t)/\mathrm{e}^{\lambda t} \to \lambda c_1$ $(t \to \infty)$. Then the function $u_1(t) = u(t)/c_1$ is a solution of (1.1) and satisfies (1.5). This finishes the proof of Theorem 1.1. $\square$

5. An example

We now give an example illustrating the main result. Consider the equation (1.1) with

$$b(t) = \frac{\sin(t^2)}{t^3} - \frac{\cos(t^2)}{t}, \quad t \geqslant t_0\,(>0). \tag{5.1}$$

Since $b(t)$ is bounded on $[t_0, \infty)$, the condition (1.8) is clearly satisfied. It is easy to see that

$$\lim_{t\to\infty} \int_{t_0}^{t} b(s)\,\mathrm{d}s = \int_{t_0}^{\infty} b(s)\,\mathrm{d}s \ \text{ exists and is finite,}$$

and

$$\int_t^\infty b(s)\,\mathrm{d}s = \frac{\sin(t^2)}{2t^2}, \quad t \geqslant t_0.$$

Then we find that

$$\begin{aligned}
\Phi_\lambda(t) &= \int_t^\infty \mathrm{e}^{-(\alpha+1)\lambda s} b(s)\,\mathrm{d}s \\
&= \mathrm{e}^{-(\alpha+1)\lambda t} \int_t^\infty b(r)\,\mathrm{d}r - (\alpha+1)\lambda \int_t^\infty \mathrm{e}^{-(\alpha+1)\lambda s}\Big(\int_s^\infty b(r)\,\mathrm{d}r\Big)\,\mathrm{d}s \\
&= \mathrm{e}^{-(\alpha+1)\lambda t}\frac{\sin(t^2)}{2t^2} - (\alpha+1)\lambda \int_t^\infty \mathrm{e}^{-(\alpha+1)\lambda s}\frac{\sin(s^2)}{2s^2}\,\mathrm{d}s, \quad t \geqslant t_0,
\end{aligned}$$

and so

$$\begin{aligned}
|\Phi_\lambda(t)| &\leqslant \mathrm{e}^{-(\alpha+1)\lambda t}\frac{1}{2t^2} + (\alpha+1)\lambda \int_t^\infty \mathrm{e}^{-(\alpha+1)\lambda s}\frac{1}{2s^2}\,\mathrm{d}s \\
&\leqslant \mathrm{e}^{-(\alpha+1)\lambda t}\frac{1}{2t^2} + \mathrm{e}^{-(\alpha+1)\lambda t}\frac{1}{2t^2} = \mathrm{e}^{-(\alpha+1)\lambda t}\frac{1}{t^2}, \quad t \geqslant t_0.
\end{aligned}$$

Then it is obvious that the condition (1.9) holds. Therefore, by Theorem 1.1, we can conclude that the equation (1.1) has a solution $u_0(t)$ satisfying (1.4) and a solution $u_1(t)$ satisfying (1.5).

The function $b(t)$ which is given by (5.1) satisfies

$$\int_{t_0}^{\infty} |b(s)| \,\mathrm{d}s = \infty. \tag{5.2}$$

To prove (5.2), note that

$$\frac{1}{2}\int_{t_0^2}^{\tau^2} \frac{|\cos\sigma|}{\sigma}\,\mathrm{d}\sigma = \int_{t_0}^{\tau} \frac{|\cos(s^2)|}{s}\,\mathrm{d}s \leqslant \int_{t_0}^{\tau} |b(s)|\,\mathrm{d}s + \int_{t_0}^{\tau} \frac{|\sin(s^2)|}{s^3}\,\mathrm{d}s \tag{5.3}$$

for $\tau \geqslant t_0$. Then, since

$$\int_{t_0^2}^{\infty} \frac{|\cos\sigma|}{\sigma}\,\mathrm{d}\sigma = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{|\sin(s^2)|}{s^3}\,\mathrm{d}s < \infty,$$

the inequality (5.3) implies (5.2). Therefore we cannot apply (i) of Theorem 1.1 in [11].

References

[1] *W. A. Coppel*: Stability and Asymptotic Behavior of Differential Equations. Heath Mathematical Monographs. Heath, Boston, 1965. zbl MR

[2] *O. Došlý, P. Řehák*: Half-Linear Differential Equations. North-Holland Mathematics Studies 202. Elsevier, Amsterdam, 2005. zbl MR doi

[3] *P. Hartman*: Ordinary Differential Equations. John Wiley, New York, 1964. zbl MR doi

[4] *J. Jaroš, K. Takaŝi, T. Tanigawa*: Nonoscillatory half-linear differential equations and generalized Karamata functions. Nonlinear Anal., Theory Methods Appl., Ser. A *64* (2006), 762–787. zbl MR doi

[5] *T. Kusano, J. Manojlović*: Precise asymptotic behavior of regularly varying solutions of second order half-linear differential equations. Electron. J. Qual. Theory Differ. Equ. *2016* (2016), Article ID 62, 24 pages. zbl MR doi

[6] *S. Luey, H. Usami*: Application of generalized Riccati equations to analysis of asymptotic forms of solutions of perturbed half-linear ordinary differential equations. Int. J. Dyn. Syst. Differ. Equ. *11* (2021), 378–390. zbl MR doi

[7] *S. Luey, H. Usami*: Asymptotic forms of solutions of perturbed half-linear ordinary differential equations. Arch. Math., Brno *57* (2021), 27–39. zbl MR doi

[8] *M. Naito*: Asymptotic behavior of nonoscillatory solutions of half-linear ordinary differential equations. Arch. Math. *116* (2021), 559–570. zbl MR doi

[9] *M. Naito*: Remarks on the existence of nonoscillatory solutions of half-linear ordinary differential equations. I. Opusc. Math. *41* (2021), 71–94. zbl MR doi

[10] *M. Naito*: Remarks on the existence of nonoscillatory solutions of half-linear ordinary differential equations. II. Arch. Math., Brno *57* (2021), 41–60. zbl MR doi

[11] *M. Naito, H. Usami*: On the existence and asymptotic behavior of solutions of half-linear ordinary differential equations. J. Differ. Equations *318* (2022), 359–383. zbl MR doi

[12] *P. Řehák*: Asymptotic formulae for solutions of half-linear differential equations. Appl. Math. Comput. *292* (2017), 165–177. zbl MR doi
[13] *P. Řehák*: Nonlinear Poincaré-Perron theorem. Appl. Math. Lett. *121* (2021), Article ID 107425, 7 pages. zbl MR doi
[14] *P. Řehák, V. Taddei*: Solutions of half-linear differential equations in the classes Gamma and Pi. Differ. Integral Equ. *29* (2016), 683–714. zbl MR doi
[15] *K. Takaŝi, J. V. Manojlović*: Asymptotic behavior of solutions of half-linear differential equations and generalized Karamata functions. Georgian Math. J. *28* (2021), 611–636. zbl MR doi

Author's address: *Manabu Naito*, Department of Mathematics, Faculty of Science, Ehime University, Matsuyama 790-8577, Japan, e-mail: `jpywm078@yahoo.co.jp`.