2023 MATHEMATICA BOHEMICA 10 pp
Online first

ON THE CLASS OF POSITIVE DISJOINT WEAK p-CONVERGENT OPERATORS

Abderrahman Retbi, Beni Mellal

Received November 28, 2022. Published online August 24, 2023.
Communicated by Laurian Suciu

Abstract. We introduce and study the disjoint weak p-convergent operators in Banach lattices, and we give a characterization of it in terms of sequences in the positive cones. As an application, we derive the domination and the duality properties of the class of positive disjoint weak p-convergent operators. Next, we examine the relationship between disjoint weak p-convergent operators and disjoint p-convergent operators. Finally, we characterize order bounded disjoint weak p-convergent operators in terms of sequences in Banach lattices.

Keywords: p-convergent operator; disjoint p-convergent operator; positive Schur property of order p; order continuous norm; Banach lattice

MSC 2020: 46A40, 46B40, 46B42

1. Introduction

In 1993 Castillo and Sánchez introduced and studied the notion of p-convergent operators on Banach spaces (see [3]). After that, many authors became interested in the study of these operators (see for instance [1], [4], [5], [10] and [12]). Recently, Zeekoei and Fourie (see [12]) considered weak and disjoint versions of the p-convergent operators, the so-called weak p-convergent operators and disjoint p-convergent operators, respectively. In this study, we continue along this path, and we introduce the disjoint version of weak p-convergent operators, the so-called disjoint weak p-convergent operators (see Definition 3.1). More precisely, we give a characterization of positive disjoint weak p-convergent operators between two Banach lattices in terms of sequences in the positive cones (see Theorem 3.2), and we derive the domination property of this class of operators (see Corollary 3.3). After that, we examine the property of indirect duality for the class of disjoint weak p-convergent operators (Theorem 3.5). Note that each disjoint p-convergent operators

DOI: 10.21136/MB.2023.0160-22

is disjoint weak p-convergent but the converse is not true in general (see Remark 3.7). Inspired by this fact, we study the relationship between disjoint weak p-convergent operators and disjoint p-convergent operators between Banach lattices (see Theorem 3.8 and Theorem 3.10). After that, we end this work with a characterization of order bounded disjoint weak p-convergent operators in terms of sequences in Banach lattices (see Theorem 3.14).

2. Definitions and notations

To state our results, we need to fix some notation and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leqslant |y|$, we have $\|x\| \leqslant \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Riesz space is said to be σ-Dedekind complete if every countable subset that is bounded above has a supremum, equivalently, whenever $0 \leqslant x_n \uparrow\leqslant x$ implies the existence of $\sup(x_n)$.

Throughout the paper we use X, Y to denote Banach spaces. The identity operator on X is denoted by Id_X and the closed unit ball of X by B_X. As is custom, we agree to use E, F to denote Banach lattices. The dual of a Banach space X is denoted by X'. We use the term operator between two Banach spaces to mean a bounded linear mapping $T\colon X \to Y$. Its adjoint operator T' is defined from Y' into X' by $T'(f)(x) = f(T(x))$ for each $f \in Y'$ and for each $x \in X$. In particular, $T\colon E \to F$ is positive if $T(x) \geqslant 0$ in F whenever $x \geqslant 0$ in E. Let $1 \leqslant p < \infty$. The conjugate number will be denoted by p', i.e., $1/p + 1/p' = 1$. The Banach space of p-summable scalar sequences is denoted by l^p and l^∞ is the space of bounded scalar sequences. The closed subspace of l^∞, consisting of the scalar sequences which are convergent with limit 0, is denoted by c_0. The unit vector basis of l^p is denoted by (e_n). Recall from [7], page 32 that a sequence (x_n) in X is weakly p-summable if $f(x_n) \in l^p$ for each $f \in X'$. A sequence (x_n) in a Banach lattice E is disjoint if $|x_n| \wedge |x_m| = 0$ for $n \neq m$.

Let $1 \leqslant p \leqslant \infty$. An operator $T\colon X \to Y$ is called

- ▷ p-convergent if T maps weakly p-summable sequences in X into norm-null sequences in Y (see [3]). The 1-convergent operators are precisely the unconditionally converging operators and the ∞-convergent operators are precisely the Dunford-Pettis operators.
- ▷ weak p-convergent if $f_n(T(x_n)) \to 0$ as $n \to \infty$ for every weakly p-summable sequence (x_n) in X, and for every weakly null sequence (f_n) in Y' (see [12]).

An operator $T\colon E \to F$ is disjoint p-convergent if it maps disjoint weakly p-summable sequences in E into norm-null sequences in F (see [12]).

A Banach space X has the Dunford-Pettis property of order p if $f_n(x_n) \to 0$ as $n \to \infty$ for every weakly p-summable sequence (x_n) in X, and for every weakly null sequence (f_n) in X' (see [3], Proposition 3.2).

A Banach lattice E has the positive Schur property of order p if each disjoint weakly p-summable sequence in E^+ is norm-null in E (see [12], Proposition 3.3).

The reader is referred to Aliprantis and Burkinshaw (see [2]), Diestel (see [6]), Diestel, Jarchow, and Tonge (see [7]) and Dunford and Schwartz (see [9]) for undefined notation and terminology.

3. Main results

We start this work by the following definition of a disjoint weak p-convergent operator between Banach lattices.

Definition 3.1. An operator T from a Banach lattice E to a Banach lattice F is disjoint weak p-convergent if $f_n(T(x_n)) \to 0$ as $n \to \infty$ for every disjoint weakly p-summable sequence (x_n) in E, and for every disjoint weakly null sequence (f_n) in F'.

Now, using the sequences in the positive cones, we characterize positive disjoint weak p-convergent operators between two Banach lattices.

Theorem 3.2. *Let E and F be two Banach lattices. For every positive operator T from E into F, the following assertions are equivalent:*

(1) *T is a disjoint weak p-convergent operator.*
(2) *For every disjoint weakly p-summable sequence $(x_n) \subset E^+$, and every disjoint weakly null sequence $(f_n) \subset (F')^+$, we have $f_n(T(x_n)) \to 0$ as $n \to \infty$.*
(3) *For every disjoint weakly p-summable sequence $(x_n) \subset E^+$, and every weakly null sequence $(f_n) \subset F'$, we have $f_n(T(x_n)) \to 0$ as $n \to \infty$.*
(4) *For every disjoint weakly p-summable sequence $(x_n) \subset E^+$, and every weakly null sequence $(f_n) \subset (F')^+$, we have $f_n(T(x_n)) \to 0$ as $n \to \infty$.*
(5) *For every weakly p-summable sequence $(x_n) \subset E^+$, and every weakly null sequence $(f_n) \subset (F')^+$, we have $f_n(T(x_n)) \to 0$ as $n \to \infty$.*

P r o o f. (1) $\Rightarrow$ (2): It is obvious.

(2) $\Rightarrow$ (3): Assume by way of contradiction that there exist a disjoint weakly p-summable sequence $(x_n) \subset E^+$, and a weakly null sequence $(f_n) \subset F'$ such that $f_n(T(x_n)) \nrightarrow 0$. The inequality $|f_n(T(x_n))| \leqslant |f_n|(T(x_n))$ implies that $|f_n|(T(x_n))$ does not converge to 0. Then there exist some $\varepsilon > 0$ and a subsequence of $|f_n|(T(x_n))$ (which we shall denote by $|f_n|(T(x_n))$ again) satisfying $|f_n|(T(x_n)) > \varepsilon$ for all n.

On the other hand, since (x_n) is a weakly p-summable sequence in E, then $(T(x_n)) \to 0$ weakly in F. Now, an easy inductive argument proves that there exist a subsequence (z_n) of (x_n) and a subsequence (g_n) of (f_n) such that

$$|g_n|(T(z_n)) > \varepsilon \quad \text{and} \quad \left(4^n \sum_{i=1}^{n} |g_i|\right)(T(z_{n+1})) < \frac{1}{n}$$

for all $n \geqslant 1$. Put $h = \sum_{i=1}^{\infty} 2^{-n}|g_n|$ and $h_n = \left(|g_{n+1}| - 4^n \sum_{i=1}^{n} |g_i| - 2^{-n}h\right)^+$. By Lemma 4.35 of [2] the sequence (h_n) is disjoint. Since $0 \leqslant h_n \leqslant |g_{n+1}|$ for all $n \geqslant 1$ and (g_n) is weakly null in F', then from Theorem 4.34 of [2], (h_n) is weakly null in F'. From the inequality

$$h_n(T(z_{n+1})) \geqslant \left(|g_{n+1}| - 4^n \sum_{i=1}^{n} |g_i| - 2^{-n}h\right)(T(z_{n+1})) \geqslant \varepsilon - \frac{1}{n} - 2^{-n}h(T(z_{n+1}))$$

we see that $h_n(T(z_{n+1})) \geqslant \frac{1}{2}\varepsilon$ must hold for all n sufficiently large (because $2^{-n}h(T(z_{n+1})) \to 0$), which contradicts our hypothesis (2).

(3) $\Rightarrow$ (4): It is obvious.

(4) $\Rightarrow$ (5): Assume by way of contradiction that there exist a weakly p-summable sequence $(x_n) \subset E^+$ and a weakly null sequence $(f_n) \subset (F')^+$ such that $f_n(T(x_n)) \nrightarrow 0$. Then there exist some $\varepsilon > 0$ and a subsequence of $f_n(T(x_n))$ (which we shall denote by $f_n(T(x_n))$ again) satisfying $f_n(T(x_n)) \geqslant \varepsilon$ for all natural numbers n.

On the other hand, as (f_n) is a weakly null sequence in (F'), we see that $(T'(f_n)) \to 0$ weakly in E'. Now, an easy inductive argument shows that there exist a subsequence (z_n) of (x_n) and a subsequence (g_n) of (f_n) such that

$$T'(g_n)(z_n) > \varepsilon \quad \text{and} \quad T'(g_{n+1})\left(4^n \sum_{i=1}^{n} z_i\right) < \frac{1}{n}$$

for all $n \geqslant 1$. Put $z = \sum_{i=1}^{\infty} 2^{-n}z_n$ and $y_n = \left(z_{n+1} - 4^n \sum_{i=1}^{n} z_i - 2^{-n}z\right)^+$. By Lemma 4.35 of [2], the sequence (y_n) is disjoint. Since $0 \leqslant y_n \leqslant z_{n+1}$ for all $n \geqslant 1$ and (z_n) is weakly p-summable in E, then from Remark 1.3 of [12], it follows that (y_n) is a weakly p-summable sequence in E. From the inequality

$$T'(g_{n+1})(y_n) \geqslant T'(g_{n+1})\left(z_{n+1} - 4^n \sum_{i=1}^{n} z_i - 2^{-n}z\right) \geqslant \varepsilon - \frac{1}{n} - 2^{-n}T'(g_{n+1})(z)$$

we see that $g_{n+1}(T(y_n)) = T'(g_{n+1})(y_n) \geqslant \frac{1}{2}\varepsilon$ must hold for all n sufficiently large (because $2^{-n}T'(g_{n+1})(z) \to 0$), which contradicts our hypothesis (4).

(5) $\Rightarrow$ (1): Let (x_n) be a weakly p-summable sequence in E consisting of pairwise disjoint terms, and let (f_n) be a weakly null sequence in F' consisting of pairwise disjoint terms. From Remark (1) of [11] we see that $(|f_n|)$ is weakly null in F', and from Proposition 2.2 of [12] we have that $(|x_n|)$ is weakly p-summable in E. So by our hypothesis (5), $|f_n|(T|x_n|) \to 0$ as $n \to \infty$. Now, from the inequality $|f_n(T(x_n))| \leqslant |f_n|(T(|x_n|))$ for all natural numbers n, we conclude that $f_n(T(x_n)) \to 0$ as $n \to \infty$, and this completes the proof. $\square$

As a consequence of Theorem 3.2, we derive the domination property for the class of positive disjoint weak p-convergent operators.

Corollary 3.3. *Let E and F be two Banach lattices. If S and T are two positive operators from E into F such that $0 \leqslant S \leqslant T$ and T is disjoint weak p-convergent, then S is also disjoint weak p-convergent.*

Proof. Let (x_n) be a weakly p-summable sequence in E^+ and (f_n) be a weak null sequence in $(F')^+$. According to (5) of Theorem 3.2, it suffices to prove that $f_n(S(x_n)) \to 0$ as $n \to \infty$. Since T is disjoint weak p-convergent, then Theorem 3.2 implies that $f_n(T(x_n)) \to 0$ as $n \to \infty$. Now, by the inequality $0 \leqslant f_n(S(x_n)) \leqslant f_n(T(x_n))$ for each natural number n, we see that $f_n(S(x_n)) \to 0$ as $n \to \infty$, and we are done. $\square$

For the proof of the next theorem, we need the following lemma, which is just Lemma 2.8 of [12].

Lemma 3.4. *Let E be a Banach lattice with type q, $1 < q \leqslant 2$, and let $p \geqslant q'$. Each disjoint sequence (x_n) in the solid hull of a relatively weakly compact subset W of E is weakly p-summable in E. In particular, the sequence $(|x_n|)$ is weakly p-summable in E.*

Now, we establish the duality property for the class of positive disjoint weak p-convergent.

Theorem 3.5. *Let T be a positive operator from a Banach lattice E into another Banach lattice F such that F' is of type q, $1 < q \leqslant 2$, and let $p \geqslant q'$. If the adjoint T' from F' into E' is disjoint weak p-convergent, then T itself is disjoint weak p-convergent.*

Proof. Let (x_n) be a disjoint weakly p-summable sequence in E^+, and let (f_n) be a disjoint weakly null sequence in $(F')^+$. Now, let $\tau\colon E \to E''$ be the canonical injection of E into its topological bidual E''. As τ is a lattice homomorphism, we see that $(\tau(x_n))$ is a weakly null sequence in $(E'')^+$. On the other hand, by Lemma 3.4

we get that (f_n) is a weakly p-summable sequence in F', and since the adjoint T' is disjoint weak p-convergent from F' into E', then from Theorem 3.2, assertion (4), we get that $(\tau(x_n))(T'(f_n)) \to 0$ as $n \to \infty$. As

$$f_n(T(x_n)) = (T'(f_n))(x_n) = (\tau(x_n))(T'(f_n))$$

holds for all natural numbers n, we deduce that $f_n(T(x_n)) \to 0$ as $n \to \infty$, and this completes the proof. □

Proposition 3.6. *Let E, F be two Banach lattices and G be a Banach space. If G has the Dunford-Pettis property of order p, then each operator $T\colon E \to F$ that admits a factorization through the Banach space G, is disjoint weak p-convergent.*

Proof. Let $P\colon E \to G$ and $Q\colon G \to F$ be two operators such that $T = Q \circ P$. Let (x_n) be a disjoint weakly p-summable sequence in E and let (f_n) be a disjoint weakly null sequence in F'. It is clear that $P(x_n)$ is weakly p-summable in G and $Q'(f_n) \to 0$ weakly in G'. As G has the Dunford-Pettis property of order p, then

$$f_n(Tx_n) = f_n(Q \circ P(x_n)) = (Q'f_n)(P(x_n)) \to 0 \quad \text{as } n \to \infty.$$

This shows that T is disjoint weak p-convergent, and this ends the proof. □

Remark 3.7. Note that each disjoint p-convergent operator is disjoint weak p-convergent, but the converse is not true in general. In fact, the identity operator $\mathrm{Id}_{c_0}\colon c_0 \to c_0$ is disjoint weak p-convergent because c_0 has the Dunford-Pettis property of order p (see Proposition 3.6), but Id_{c_0} is not disjoint p-convergent because (e_n) is a disjoint weak p-summable sequence in c_0 and $\|e_n\| \nrightarrow 0$ as $n \to \infty$.

Now, we are in a position to give the relation between disjoint weak p-convergent and disjoint p-convergent operators.

Theorem 3.8. *Let E and F be two Banach lattices such that F is σ-Dedekind complete. If each positive disjoint weak p-convergent operator $T\colon E \to F$ is disjoint p-convergent, then one of the following assertions is valid:*

(1) *E has the positive Schur property of order p,*
(2) *the norm of F is order continuous.*

Proof. Assume by way of contradiction that E does not have positive Schur property of order p and F does not have the order continuous norm. We have to construct a positive disjoint weak p-convergent operator which is not disjoint p-convergent.

Since E does not have the positive Schur property of order p, there exists a disjoint weakly p-summable sequence (x_n) in E^+ which is not norm convergent to 0. As $\|x_n\| = \sup\{|f(x_n)|\colon f \in (E')^+, \|f\| = 1\}$, there exist a sequence $f_n \in (E')^+$ with $\|f_n\| = 1$, some $\varepsilon > 0$ and a subsequence (y_n) of (x_n) such that $|f_n(y_n)| = f_n(y_n) \geqslant \varepsilon$ for all natural numbers n. Now, consider the operator $P\colon E \to l^\infty$ defined by

$$P(x) = (f_k(x))_{k=1}^\infty.$$

Clearly, P is positive. Also, since the norm of the Dedekind σ-complete Banach lattice F is not order continuous, it follows from Theorem 4.51 of [2] that l^∞ is lattice embeddable in F. Let $Q\colon l^\infty \to F$ be a lattice embedding. Then there exist $m > 0$ and $M > 0$ such that

$$m\|((\lambda_k)_{k=1}^\infty)\|_\infty \leqslant \|Q((\lambda_k)_{k=1}^\infty)\| \leqslant M\|((\lambda_k)_{k=1}^\infty)\|_\infty$$

for all $((\lambda_k)_{k=1}^\infty) \in l^\infty$. Note that Q is also a lattice homomorphism, and hence it is positive (see page 235 and 236 of [2]). Let $T = Q \circ P\colon E \to l^\infty \to F$. As l^∞ has the Dunford-Pettis property of order p, it follows from Proposition 3.6 that T is a positive disjoint weak p-convergent operator. On the other hand, T is not disjoint p-convergent. In fact, note that (y_n) is a disjoint weakly p-summable sequence in E and for every $n \in \mathbb{N}$ we have

$$\|T(y_n)\| = \|Q((f_k(y_n))_{k=1}^\infty)\|_\infty \geqslant m\|(f_k(y_n))_{k=1}^\infty\|_\infty \geqslant mf_n(y_n) \geqslant m\varepsilon.$$

This shows that T is not disjoint p-convergent, and we are done. □

Remark 3.9. The second necessary condition of Theorem 3.8 is not sufficient. In fact, the identity operator $\mathrm{Id}_{c_0}\colon c_0 \to c_0$ is disjoint weak p-convergent but is not disjoint p-convergent. However, the norm of c_0 is order continuous. Now, if we replace the arrival space F by its topological dual space F', then the second necessary condition of Theorem 3.8 becomes sufficient; for more details see the next theorem.

Theorem 3.10. *Let E and F be two Banach lattices. Then the following assertions are equivalent:*

(1) *Each positive disjoint weak p-convergent operator $T\colon E \to F'$ is disjoint p-convergent.*
(2) *One of the following assertions is valid:*
    (a) *E has the positive Schur property of order p,*
    (b) *the norm of F' is order continuous.*

P r o o f. (1) ⇒ (2): It follows from Theorem 3.8.

(2a) ⇒ (1): It is obvious.

(2b) ⇒ (1): Let (x_n) be a disjoint weakly p-summable sequence in E^+. Then $|T(x_n)| = T(x_n)$ is weak* null sequence in $(F')^+$. Now, let (y_n) be a disjoint norm bounded sequence in F^+. As the norm of F' is order continuous, it follows from Corollary 2.9 of [8] that (y_n) is weakly null in F^+. Since the canonical injection $\tau\colon F \to F''$ is a lattice homomorphism, we see that $\tau(y_n)$ is a disjoint weakly null sequence in $(F'')^+$. As $T\colon E \to F'$ is a positive disjoint weak p-convergent operator, it follows from Theorem 3.2, assertion (2), that $(\tau(y_n))(T(x_n)) \to 0$ as $n \to \infty$. The equality

$$(\tau(y_n))(T(x_n)) = (T(x_n))(y_n)$$

implies that $(T(x_n))(y_n) \to 0$ as $n \to \infty$. By Corollary 2.7 of [8] we conclude that $\|T(x_n)\| \to 0$ as $n \to \infty$, and the proof is complete. □

From Theorem 1.18 of [2] and Proposition 2.2 of [12], we obtain:

Proposition 3.11. *Let E be a Banach lattice and (f_n) be a disjoint weakly p-summable sequence in E'. Then for each $z \in E^+$ we have*

$$\sup_{x\in[-z,z]} |f_n(x)| = |f_n|(z) \to 0 \quad \text{as } n \to \infty.$$

As a simple consequence we derive:

Corollary 3.12. *Let $T\colon E \to F$ be an order bounded operator between two Banach lattices E and F. If (f_n) is a disjoint weakly p-summable sequence in F', then for each $z \in E^+$ we have*

$$\sup_{y\in T([-z,z])} |f_n(y)| = |f_n \circ T|(z) \to 0 \quad \text{as } n \to \infty.$$

Proposition 3.13. *Let $T\colon E \to F$ be an order bounded operator from a Banach lattice E into another Banach lattice F, and let A be a norm bounded solid subset of E. The following statements are equivalent:*

(1) *For every disjoint weakly p-summable sequence (f_n) in F' we have*

$$\sup_{y\in T(A)} |f_n(y)| \to 0 \quad \text{as } n \to \infty.$$

(2) *For every disjoint sequence (x_n) in $A^+ = A \cap E^+$ and for every disjoint weakly p-summable sequence (f_n) in F' we have $\sup_{y\in\{T(x_n)\colon n\in\mathbb{N}\}} |f_n(y)| \to 0$ as $n \to \infty$.*

(3) *For every disjoint sequence (x_n) in A^+ and for every disjoint weakly p-summable sequence (f_n) in F' we have $f_n(T(x_n)) \to 0$ as $n \to \infty$.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1): As $\sup_{y\in T(A)}|f_n(y)| = \sup_{x\in A}|f_n(T(x))|$, then it suffices to show that $\sup_{x\in A}|f_n(T(x))| \to 0$ as $n\to\infty$ for every weakly p-summable sequence (f_n) in F'. Otherwise, there exists a sequence (f_n) in F' satisfying $\sup_{x\in A}|f_n(T(x))| > \varepsilon$ for some $\varepsilon > 0$ and all $n\in\mathbb{N}$. Hence, for every natural number n there exists $z_n \in A^+$ such that $|T'(f_n)|(z_n) > \varepsilon$. On the other hand, it follows from Corollary 3.12 that $|f_n\circ T|(z) \to 0$ as $n\to\infty$ for each $z\in E^+$. An inductive argument proves that there exist a subsequence (y_n) of (z_n) and a subsequence (g_n) of (f_n) such that

$$|T'(g_{n+1})|(y_{n+1}) > \varepsilon \quad\text{and}\quad |T'(g_{n+1})|\biggl(4^n\sum_{i=1}^{n} y_i\biggr) < \frac{1}{n}$$

for all $n\geqslant 1$. Now, put $x = \sum\limits_{i=1}^{\infty} 2^{-i}y_i$ and $x_n = \Bigl(y_{n+1} - 4^n\sum\limits_{i=1}^{n} y_i - 2^{-n}x\Bigr)^+$. By Lemma 4.35 of [2] the sequence (x_n) is disjoint. As $0\leqslant x_n\leqslant y_{n+1}$ for all $n\geqslant 1$ and (y_{n+1}) in A^+, we see that (x_n) in A^+. From the inequality,

$$|T'(g_{n+1})|(x_n) \geqslant T'(g_{n+1})\biggl(y_{n+1} - 4^n\sum_{i=1}^{n} y_i - 2^{-n}x\biggr) \geqslant \varepsilon - \frac{1}{n} - 2^{-n}|T'(g_{n+1})|(x).$$

Hence, $|T'(g_{n+1})|(x_n) \geqslant \frac{1}{2}\varepsilon$ must hold for all n sufficiently large (because $2^{-n}\times T'(g_{n+1})(x) \to 0$ as $n\to\infty$).

In view of $|T'(g_{n+1})|(x_n) = \sup\{|g_{n+1}(T(z))|\colon |z|\leqslant x_n\}$, for each n sufficiently large there exists some $|z_n|\leqslant x_n$ with $|g_{n+1}(T(z_n))| > \frac{1}{2}\varepsilon$. As (z_n^+) and (z_n^-) are norm bounded disjoint sequences in A^+, by our hypothesis we see that

$$\frac{\varepsilon}{2} < |g_{n+1}(T(z_n))| \leqslant |g_{n+1}(T(z_n^+))| + |g_{n+1}(T(z_n^-))| \to 0$$

as $n\to\infty$, a contradiction, and this completes the proof. □

Theorem 3.14. *Let E and F be two Banach lattices such that E is of type q, $1<q\leqslant 2$, and F' is of type r, $1<r\leqslant 2$, and let $p\geqslant \max(q',r')$. Let T be an order bounded operator from E into F. Then the following assertions are equivalent:*

(1) *T is a disjoint weak p-convergent operator.*
(2) *$f_n(T(x_n)) \to 0$ as $n\to\infty$ for every disjoint weakly p-summable sequence (x_n) in E and for every disjoint weakly null sequence (f_n) in F'.*
(3) *$f_n(T(x_n)) \to 0$ as $n\to\infty$ for every weakly p-summable sequence (x_n) in E and for every disjoint weakly null sequence (f_n) in F'.*

P r o o f. (1) $\Leftrightarrow$ (2): It follows from Definition 3.1.

(3) $\Rightarrow$ (2): It is Obvious.

(2) $\Rightarrow$ (3): Let (x_n) be a weakly p-summable sequence in E and (f_n) be a disjoint weakly null sequence in F'. We show that $f_n(T(x_n)) \to 0$ as $n \to \infty$. Let A be the solid hull of the weak relatively compact subset $\{x_n : n \in \mathbb{N}\}$ of E. If (z_n) is a disjoint sequence in A^+, then from Lemma 3.4 we have that (z_n) is a weakly p-summable sequence in E (because E is of type q, $1 < q \leqslant 2$, and $p \geqslant q'$), and by our hypothesis we get $f_n(T(z_n)) \to 0$ as $n \to \infty$. As F' is of type r, $1 < r \leqslant 2$, and $p \geqslant r'$, then by Lemma 3.4 we have that (f_n) is disjoint weakly p-summable in F'. Now, Proposition 3.13 implies that $\sup_{y \in T(A)} |f_n(y)| \to 0$ as $n \to \infty$. Since

$$|f_n(T(x_n))| \leqslant \sup_{x \in A} |f_n(T(x))| \leqslant \sup_{y \in T(A)} |f_n(y)|,$$

we obtain $f_n(T(x_n)) \to 0$ as $n \to \infty$, and the proof is finished. □

A c k n o w l e d g m e n t. The author is thankful to the referee for their valuable comments and suggestions.

References

[1] *M. Alikhani, M. Fakhar, J. Zafarani*: p-convergent operators and the p-Schur property. Anal. Math. *46* (2020), 1–12. zbl MR doi

[2] *C. D. Aliprantis, O. Burkinshaw*: Positive Operators. Springer, Dordrecht, 2006. zbl MR doi

[3] *J. M. F. Castillo, F. Sánchez*: Dunford-Pettis-like properties of continuous vector function spaces. Rev. Mat. Univ. Complutense Madr. *6* (1993), 43–59. zbl MR

[4] *D. Chen, J. A. Chávez-Domínguez, L. Li*: p-converging operators and Dunford-Pettis property of order p. J. Math. Anal. Appl. *461* (2018), 1053–1066. zbl MR doi

[5] *M. B. Dehghani, S. M. Moshtaghioun*: On the p-Schur property of Banach spaces. Ann. Funct. Anal. *9* (2018), 123–136. zbl MR doi

[6] *J. Diestel*: Sequences and Series in Banach Spaces. Graduate Texts in Mathematics 92. Springer, New York, 1984. zbl MR doi

[7] *J. Diestel, H. Jarchow, A. Tonge*: Absolutely Summing Operators. Cambridge Studies in Advanced Mathematics 43. Cambridge University Press, Cambridge, 1995. zbl MR doi

[8] *P. G. Dodds, D. H. Fremlin*: Compact operators in Banach lattices. Isr. J. Math. *34* (1979), 287–320. zbl MR doi

[9] *N. Dunford, J. T. Schwartz*: Linear Operators. I. General Theory. Pure and Applied Mathematics 7. Interscience Publishers, New York, 1958. zbl MR

[10] *I. Ghenciu*: The p-Gelfand-Phillips property in space of operators and Dunford-Pettis like sets. Acta Math. Hung. *155* (2018), 439–457. zbl MR doi

[11] *W. Wnuk*: Banach lattices with the weak Dunford-Pettis property. Atti Semin. Mat. Fis. Univ. Modena *42* (1994), 227–236. zbl MR

[12] *E. D. Zeekoei, J. H. Fourie*: On p-convergent operators on Banach lattices. Acta Math. Sin., Engl. Ser. *34* (2018), 873–890. zbl MR doi

Author's address: *Abderrahman Retbi*, Polydisciplinary Faculty, Sultan Moulay Slimane University, B.P. 592, Mghila, Beni Mellal, Morocco, e-mail: `abderrahmanretbi@gmail.com`.