REDUCING THE LENGTHS OF SLIM PLANAR SEMIMODULAR LATTICES WITHOUT CHANGING THEIR CONGRUENCE LATTICES

Gábor Czédli, Szeged

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Dedicated to the memory of my paternal grandfather József

Abstract. Following G. Grätzer and E. Knapp (2007), a slim planar semimodular lattice, SPS lattice for short, is a finite planar semimodular lattice having no M_3 as a sublattice. An SPS lattice is a slim rectangular lattice if it has exactly two doubly irreducible elements and these two elements are complements of each other. A finite poset P is said to be JConSPS-representable if there is an SPS lattice L such that P is isomorphic to the poset $\mathrm{J}(\mathrm{Con}L)$ of join-irreducible congruences of L. We prove that if $1 < n \in \mathbb{N}$ and P is an n-element JConSPS-representable poset, then there exists a slim rectangular lattice L such that $\mathrm{J}(\mathrm{Con}L) \cong P$, the length of L is at most $2n^2$, and $|L| \leqslant 4n^4$. This offers an algorithm to decide whether a finite poset P is JConSPS-representable (or a finite distributive lattice is "ConSPS-representable"). This algorithm is slow as G. Czédli, T. Dékány, G. Gyenizse, and J. Kulin proved in 2016 that there are asymptotically $\frac{1}{2}(k-2)!\,\mathrm{e}^2$ slim rectangular lattices of a given length k, where e is the famous constant ≈ 2.71828. The known properties and constructions of JConSPS-representable posets can accelerate the algorithm; we present a new construction.

Keywords: slim rectangular lattice; slim semimodular lattice; planar semimodular lattice; congruence lattice; lattice congruence; lamp; $\mathcal{C}_1$-diagram

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1. Introduction

Following Grätzer and Knapp (see [20]), a *slim planar semimodular lattice*, *SPS lattice* for short, is a finite planar (upper) semimodular lattice having no M_3 as

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a sublattice. By Grätzer and Knapp (see [21]), an SPS lattice L is a *slim rectangular lattice* if it has exactly two doubly irreducible elements (denoted by $\mathrm{lc}(L)$ and $\mathrm{rc}(L)$ and called the *left corner* and the *right corner* of L) and these two elements are complements of each other. As usual, $\mathrm{J}(L)$, the set of join-irreducible elements, is $\{x \in L\colon x \text{ has exactly one lower cover}\}$; $\mathrm{M}(L)$ is defined dually. As in Czédli and Schmidt (see [16]), a lattice L is *slim* if it is finite and $\mathrm{J}(L)$ is the union of two chains. We know from Czédli and Schmidt (see [16], Lemma 2.3) that for a lattice L,

$$L \text{ is an SPS lattice} \Leftrightarrow L \text{ is a slim semimodular lattice.} \tag{1.1}$$

In the paper as in many earlier ones, "slim semimodular" means the same as "slim planar semimodular", that is, "SPS". A finite lattice D is *ConSPS-representable* if it is isomorphic to the congruence lattice $\mathrm{Con}\, L$ of an SPS lattice L. Similarly, a finite poset P is *JConSPS-representable* if $P \cong \mathrm{J}(\mathrm{Con}\, L)$ for an SPS lattice L.

Due to (historical) Section 2 in Czédli and Kurusa (see [14]), the surveying part of this section is reduced to a few comments. The four dozen element list[1] in Appendix of Czédli (see `https://arxiv.org/abs/2107.10202`) shows that since 2007, SPS lattices form an intensively investigated class of lattices. In addition to their impact on and connection with geometry, group theory, and combinatorics as explained in [14], SPS lattices have connections with finite model theory, see Czédli [9]. SPS lattices (or their duals) are particular cases of some other classes of lattices and combinatorial structures; indeed, they are also join-distributive lattices, meet-semidistributive lattices, and subspace lattices of antimatroids (or convex geometries); see, for example, Czédli [4]. Thus, benefiting from the fact that SPS lattices are well understood by means of several structure theorems and representation theorems, the study of these lattices can lead to discoveries for larger classes of lattices and related structures; for example, see Adaricheva and Czédli [1]. Actually, even purely geometric papers are in connection with SPS lattices; see, for example, Czédli and Kurusa [14]. By Grätzer and Knapp (see [20], Section 3), the theory of planar semimodular lattices is satisfactorily reduced to that of SPS lattices. So last (and least) we note that there are some problems where it could be possible or it was possible to prove more for planar semimodular or SPS lattice than for all finite lattices; see, e.g., Ahmed and Horváth [2], and Czédli and Schmidt [15].

Within lattice theory, the interest in SPS lattices is mainly fueled by (see Grätzer [18], Problem 1) asking for a characterization of ConSPS-representable distributive lattices. Note that Problem 1 of [18] is motivated by the fact that M_3 sublattices played a key role in Grätzer, Lakser, and Schmidt [22] representing all finite distribu-

[1] See `http://www.math.u-szeged.hu/~czedli/m/listak/publ-psml.pdf` for an update.

tive lattices by congruence lattices of planar semimodular lattices, whereby it was natural to ask what happens when M_3 sublattices are not permitted, that is, when SPS rather than planar semimodular lattices are used.

Since ConSPS-representability implies distributivity and a finite distributive lattice D is perfectly described by $\mathrm{J}(D)$, a satisfactory characterization of JConSPS-representable posets would yield a characterization of ConSPS-representable lattices. However, the two representability problems are not the same in the aspect of axiomatizability. Indeed, Czédli in [9] proves that JConSPS-representable posets cannot be described by finitely many axioms in the first-order language of finite posets but it is still unknown whether ConSPS-representable lattices have a finite first-order axiomatization in the class of finite lattices. Note that the class of JConSPS representable posets has many known properties and is closed under some constructions; see Remark 6.3 for bibliographic details. However, we do not know whether these properties and constructions themselves offer an algorithm to decide whether a poset is JConSPS-representable or not. Indeed, since we do not know whether the collection of the above-mentioned known properties and constructions is sound and even a very large SPS lattice can JConSPS-represent a small poset[2] P, it is not clear at first sight whether it suffices to check $\mathrm{J}(\mathrm{Con}\, L)$ for finitely many L.

2. Goal

In Theorem 5.1, we give an upper bound on the length of the shortest slim rectangular lattices L that JConSPS-represents a given JConSPS-representable finite poset P. Therefore, there exists an algorithm to decide if a finite poset P is JConSPS-representable; indeed, we know from Czédli, Dékány, Gyenizse, and Kulin (see [12]) that, up to isomorphism,

$$\left\{\begin{array}{l}\text{the number of slim rectangular lattices of a given length } k\\ \text{is asymptotically } \frac{1}{2}(k-2)!\,\mathrm{e}^2, \text{ where } \mathrm{e} = \lim\limits_{n\to\infty}(1+1/n)^n \approx 2.71828.\end{array}\right. \tag{2.1}$$

By (2.1), there are only finitely many slim rectangular lattices up to a given length. Thus, Theorem 5.1 implies the existence of an algorithm that for each finite poset P decides whether P is JConSPS-representable. Moreover, if P is such and $|P| > 1$, then the algorithm constructs a slim rectangular lattice L such that $P \cong \mathrm{J}(\mathrm{Con}\, L)$. Remark 6.3 points out that known properties and constructions, including the multifork extension construction, make the algorithm faster. Proposition 6.1 presents a new construction that extends a JConSPS-representable poset to a larger one.

[2] E.g., with $\mathsf{S}_7^{(1)}, \mathsf{S}_7^{(2)}, \dots$ in Figure 2, we have that $|\mathrm{J}(\mathrm{Con}\,\mathsf{S}_7^{(k)})| = 5$ for all (large) k.

3. Concepts, terminology, and tools from earlier papers

As in Czédli [7] and thereafter, to avoid subscripts of subscripts, the bottom 0_I and the top 1_I of an interval I are denoted by $\mathrm{Foot}(I)$ and $\mathrm{Peak}(I)$, respectively. For u in a lattice L, $\downarrow u = \downarrow_L u := \{x \in L\colon\ x \leqslant u\}$ and $\uparrow u = \uparrow_L u := \{x \in L\colon\ x \geqslant u\}$. Edges in a planar diagram are *straight* line segments denoting prime intervals $\mathfrak{p} = [\mathrm{Foot}(\mathfrak{p}), \mathrm{Peak}(\mathfrak{p})]$. A usual coordinate system of the plane is always fixed. Edges (or lines) parallel to $(1,1)$ or $(1,-1)$ are of *normal slopes.* Edges parallel to $(1,t)$ for some $t \in \mathbb{R}$ with $|t| > 1$ and vertical edges are said to be *precipitous.*

Going after Grätzer and Knapp (see [20] and [21]), let $L^\sharp$ be a planar diagram of a slim rectangular lattice L. The *left boundary chain* and the *right boundary chain* of $L^\sharp$ are denoted by $\mathrm{LBnd}(L)$ and $\mathrm{RBnd}(L)$, respectively. (Actually, $\mathrm{LBnd}(L^\sharp)$ and $\mathrm{RBnd}(L^\sharp)$ would be more precise but we always fix $L^\sharp$ in a way to be defined soon. This comment applies for several other concepts we are going to define.) The *boundary* of L is $\mathrm{Bnd}(L) = \mathrm{LBnd}(L) \cup \mathrm{RBnd}(L)$. The elements of $\mathrm{Bnd}(L)$ and those of $L \setminus \mathrm{Bnd}(L)$ are called *boundary elements* and *internal elements.* For example, the already mentioned corners are boundary elements: $\mathrm{lc}(L) \in \mathrm{LBnd}(L)$ and $\mathrm{rc}(L) \in \mathrm{RBnd}(L)$. For $x \in L$, the *left support* and the *right support* of x are[3]

$$(3.1)\qquad \begin{cases} \mathrm{lsupp}(x) := x \wedge \mathrm{lc}(L) \text{ and } \mathrm{rsupp}(x) := x \wedge \mathrm{rc}(L).\\ \text{Note that } x = \mathrm{lsupp}(x) \vee \mathrm{rsupp}(x),\\ \mathrm{lsupp}(x) \text{ is on the \textit{lower left boundary} } \downarrow_L \mathrm{lc}(L),\ \downarrow_L \mathrm{lc}(L) \subseteq \mathrm{LBnd}(L),\\ \mathrm{rsupp}(x) \text{ is on the \textit{lower right boundary} } \downarrow_L \mathrm{rc}(L),\ \downarrow_L \mathrm{rc}(L) \subseteq \mathrm{RBnd}(L). \end{cases}$$

The *upper left boundary* and the *upper right boundary* of L are the principal filters $\uparrow_L \mathrm{lc}(L)$ and $\uparrow_L \mathrm{rc}(L)$; note that $\uparrow_L \mathrm{lc}(L) \subseteq \mathrm{LBnd}(L)$ and $\uparrow_L \mathrm{rc}(L) \subseteq \mathrm{RBnd}(L)$.

Recall from Czédli [7], Definition 2.1 (as Czédli [6] would be too general here) that the diagram $L^\sharp$ of L is a *$\mathcal{C}_1$-diagram* if for every edge $\mathfrak{p} = [\mathrm{Foot}(\mathfrak{p}), \mathrm{Peak}(\mathfrak{p})]$ of the diagram, $\mathfrak{p}$ is either precipitous or it is of a normal slope and, furthermore, $\mathfrak{p}$ is precipitous $\Leftrightarrow$ $\mathrm{Foot}(\mathfrak{p})$ is an internal meet-irreducible element of L.

Convention 3.1. Together with each slim rectangular lattice occurring in the paper, a $\mathcal{C}_1$-diagram of our lattice is fixed. Moreover, even if we do not say it all the time, whenever we construct a lattice (like a sublattice or a larger lattice), then we always construct its fixed $\mathcal{C}_1$-diagram as well. In notation, we rarely distinguish a slim rectangular lattice from its $\mathcal{C}_1$-diagram.

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[3] The third equality in (3.1) follows from (1.1) and Grätzer and Knapp, see [21], Lemma 3 and Lemma 4.

Complying with Convention 3.1, all lattice diagrams in this paper are $\mathcal{C}_1$-diagrams. Let L denote a slim rectangular lattice. Note in advance that quite often,

(3.2) we do not distinguish between lattice theoretic and geometric objects.

If $a < b \in L$ and C_1, C_2 are maximal chains of the interval $[a, b]$ such that $C_1 \cap C_2 = \{a, b\}$ and all elements x of C_1 are on the left of C_2 (including the possibility of $x \in C_2$), then the elements $[a, b]$ that are simultaneously on the right of C_1 and on the left of C_2 form a so-called *lattice region*; see Kelly and Rival [23] for a more exact definition. The corresponding geometric area, which is bordered by C_1 and C_2, is a *geometric region*. Note that whenever we define a geometric area (like a geometric region) or a line segment, then (unless otherwise explicitly stated) it contains its boundaries, that is, it is *topologically closed*. Minimal non-chain regions are *cells*. If a cell contains exactly four lattice elements, then it is a 4-*cell*. Note that 4-cells are cover-preserving boolean sublattices with 4 elements but, as M_3 exemplifies, not conversely. A 4-*cell lattice* is a planar lattice in which all cells are 4-cells (in a fixed planar diagram). Grätzer and Knapp in [20], Lemmas 4–5, and in [21] proved that for a planar lattice L (which is finite by definition),

$$(3.3)\qquad \begin{cases} \text{if } L \text{ is a 4-cell lattice, no two distinct 4-cells have the same bottom,} \\ L \text{ has exactly two doubly irreducible elements, and these two elements} \\ \text{are complementary, then } L \text{ is a slim rectangular lattice. Conversely,} \\ \text{every slim rectangular lattice is a 4-cell lattice with these properties.} \end{cases}$$

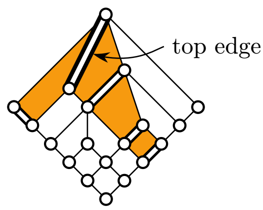

Figure 1. A trajectory.

On the set of prime intervals (i.e., edges) of a slim rectangular lattice L, let τ be the smallest equivalence relation that collapses the opposite sides of every 4-cell. As in Czédli and Schmidt [16], the blocks of τ are called *trajectories*; e.g., the double-lined edges form a trajectory in Figure 1. Going from left to right, a trajectory does not branch out and neither it does so backwards. The unique edge $\mathfrak{p}$ of a trajectory such that $\mathrm{Foot}(\mathfrak{p}) \in \mathrm{M}(L)$ is the *top edge* of the trajectory. The *ascending part* of a trajectory consists of the top edge and all of its edges left to the top edge; the *descending part* is defined left-right symmetrically. Any two consecutive edges of a trajectory form a 4-*cell of a the trajectory*; they are orange-filled in the figure.

Given a 4-cell H of L and a positive integer $k \in \mathbb{N}^+$, we obtain the k-fold *multifork extension* of L at H by changing H to a copy of $\mathsf{S}_7^{(k)}$ and proceeding to the southeast and to the southwest to preserve semimodularity. For the exact definition, see Czédli [5], where this construction was introduced, or see Figure 2, where the construction is illustrated by performing a 1-fold multifork extension at H_1 of L_0 to obtain L_1 and performing a 3-fold multifork extension at H_2 of L_1 to obtain L_2. (To save space, our figures are multi-purpose figures; some ingredients of Figure 2 are explained later.) Note in advance that the thick edges of our lattice diagrams are called neon tubes. Note also that 1-fold multifork extensions are also called *fork extensions*; see Czédli and Schmidt [17]; in this case the new elements form a so-called *fork* in the new lattice; see (4.1) later.

A *grid* is (the fixed $\mathcal{C}_1$-diagram of) the direct product of two non-singleton finite chains. A 4-cell H of L is a *distributive* 4-*cell* if the principal ideal $\downarrow_L \mathrm{Peak}(H)$ is a distributive lattice. By Czédli and Schmidt [17] and the following lemma,

(3.4) if H is a distributive 4-cell of L, then $\downarrow_L \mathrm{Peak}(H)$ is a grid.

The most useful structure theorem of slim rectangular lattices is the following.

Lemma 3.2 (Multifork Sequence Lemma [5], Theorem 3.7)**.** *For each slim rectangular lattice L, there exist positive integers $m_1, \dots, m_k$, a sequence $L_0, L_1, \dots, L_k$ of slim rectangular lattices, and a distributive 4-cell H_i of L_{i-1} for $i \in \{1, \dots, k\}$ such that L_0 is a grid, $L_k = L$, and L_i is obtained from L_{i-1} by performing an m_i-fold multifork extension at H_i for $i \in \{1, \dots, k\}$. Furthermore, any lattice obtained in this way from a grid is a slim rectangular lattice.*

The system $(L_0, H_1, m_1, L_1, H_2, m_2, \dots, L_{k-1}, H_k, m_k, L = L_k)$ with components as above is the *multifork sequence of L*; it is not necessarily unique but we always fix one. (Note, however, that k is unique.)

Definition 3.3 (Czédli [10])**.** Let $\mathfrak{n}$ be an edge on the upper boundary of the initial grid L_0. The union of the 4-cells of the trajectory containing $\mathfrak{n}$ is the *original territory* of $\mathfrak{n}$; it is denoted by $\mathrm{OT}(\mathfrak{n})$. When we obtain L_i from L_{i-1}, then we add several new edges and exactly m_i of these new edges have the same peak as H. Let $\mathfrak{n}$ be one of these new edges. In L_i, the union of the 4-cells of the trajectory containing $\mathfrak{n}$ is a geometric area; we call it the *original territory* $\mathrm{OT}(\mathfrak{n})$ of $\mathfrak{n}$ in L. Note that we have defined $\mathrm{OT}(\mathfrak{n})$ if and only if $\mathfrak{n}$ is an edge of the upper boundary or $\mathfrak{n}$ is a precipitous edge. If $\mathfrak{n}$ is an edge of the upper left boundary chain, then the essential part of the original territory, denoted by $\mathrm{EOT}(\mathfrak{n})$, and the right essential part of the original territory, denoted by $\mathrm{REOT}(\mathfrak{n})$, of $\mathfrak{n}$ are $\mathrm{OT}(\mathfrak{n})$ while the left essential part of the original territory, denoted by $\mathrm{LEOT}(\mathfrak{n})$, of $\mathfrak{n}$ is the empty set. Similarly, for $\mathfrak{n}$ on the right upper boundary, $\mathrm{EOT}(\mathfrak{n}) = \mathrm{LEOT}(\mathfrak{n}) := \mathrm{OT}(\mathfrak{n})$ and $\mathrm{REOT}(\mathfrak{n}) = \emptyset$. Next, let $\mathfrak{n}$ be a precipitous new edge of L_i and denote by T the trajectory of L_i that contains $\mathfrak{n}$. The union of the 4-cells of T that do not contain $\mathfrak{n}$ as an edge is the essential part $\mathrm{EOT}(\mathfrak{n})$ of the original territory of $\mathfrak{n}$; it is a geometric area and the union of two (geometrically) connected subsets that are, in a self-explanatory manner, called the *left essential part* $\mathrm{LEOT}(\mathfrak{n})$ and the *right essential part* $\mathrm{REOT}(\mathfrak{n})$ *of* the *original territory* of $\mathfrak{n}$.

For examples of $\mathrm{OT}(\mathfrak{n}), \dots, \mathrm{REOT}(\mathfrak{n})$, see Figures 2, 3, and 4. Even though their definition relies on L_0 or L_i, we also use these concepts in L, where $\mathrm{OT}(\mathfrak{n}), \dots,$ $\mathrm{REOT}(\mathfrak{n})$ have no connection with the trajectory containing $\mathfrak{n}$ in general; this is exemplified by $\mathfrak{n}_1$ and $\mathfrak{n}_2$ in L' (but not in L) of Figure 3. Statement (3.4) implies that

$$(3.5)\quad \begin{cases} \text{if } \mathrm{OT}(\mathfrak{n}) \text{ is defined, then it is bordered by edges of } L \text{ and all of these} \\ \text{edges with peaks different from } \mathrm{Peak}(\mathfrak{n}) \text{ are of normal slopes.} \\ \text{Furthermore, each of } \mathrm{LEOT}(\mathfrak{n}) \text{ and } \mathrm{REOT}(\mathfrak{n}) \text{ is either the empty set} \\ \text{or a rectangle bordered by edges of normal slopes. See also (3.6) later.} \end{cases}$$

Definition 3.4 (Czédli [7])**.** Let L be a slim rectangular lattice.

(A) The prime intervals $\mathfrak{p}$ of L with $\mathrm{Foot}(\mathfrak{p}) \in \mathrm{M}(L)$ are called *neon tubes*. If $\mathrm{Foot}(\mathfrak{p}) \in \mathrm{Bnd}(L)$, then $\mathfrak{p}$ is a *boundary neon tube* and it is of a normal slope. Otherwise, $\mathfrak{p}$ is an *internal neon tube* and it is precipitous. (Convention 3.1 applies.)

(B) Boundary lamps are the same as boundary neon tubes. (However, if $I = \mathfrak{p}$ is a boundary lamp, then we sometimes say that $\mathfrak{p}$ is the neon tube of I.) An interval I is an *internal lamp* if $\mathrm{Peak}(I)$ is the peak of an internal neon tube and $\mathrm{Foot}(I)$ is the meet of the feet of all internal neon tubes with the peak $\mathrm{Peak}(I)$. (These neon tubes are called the neon tubes of I.)

(C) In our lattice diagrams (which are $\mathcal{C}_1$-diagrams), the neon tubes are exactly the thick edges and the feet of the lamps are black-filled. We know from Czédli [7], Lemma 3.1 that a lamp is uniquely determined by its foot. Thus, for a lamp I, we label the black-filled vertex $\mathrm{Foot}(I)$ in our figures by I rather than by $\mathrm{Foot}(I)$.

Lamps have been the fundamental tool to study JConSPS-representability in Czédli [7], [9], [11], [10], and Czédli and Grätzer [13]. Lamps are particular intervals I. Sometimes, we need to consider them pairs $(\mathrm{Foot}(I), \mathrm{Peak}(I))$. The (geometric) rectangle bordered by $\mathrm{LBnd}(L)$ and $\mathrm{RBnd}(L)$ is the *full geometric rectangle* $\mathrm{FullRect}(L)$ of L. Combining Definition 3.3 with Czédli [7], recall the following.

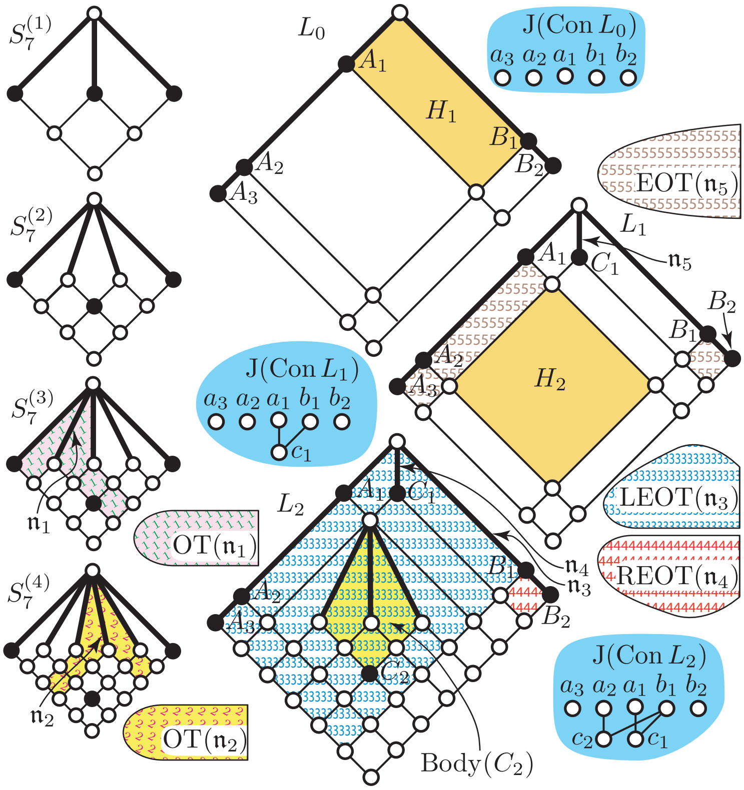

Figure 2. Multifork extensions and some geometric objects.

Definition 3.5 (Some geometric areas and polygons; Czédli [7])**.** For a slim rectangular lattice (diagram) L, let K be an interval, I and J be lamps, and $\mathfrak{p}$ be a neon tube of L.

(A) The illuminated area $\mathrm{Lit}(I)$ of I is the union of the original territories of the neon tubes of I.

(B) The *left roof* and the *left floor* of the interval K of L are the line segments of slope $(1,1)$ with lower endpoints on the left boundary chain and upper endpoints $\mathrm{Peak}(K)$ and $\mathrm{Foot}(K)$, respectively. They are denoted by $\mathrm{LRoof}(K)$ and $\mathrm{LFloor}(K)$, respectively. With slope $(1,-1)$, the *right roof* $\mathrm{RRoof}(K)$ and the *right floor* $\mathrm{RFloor}(K)$ are defined analogously. The *roof* $\mathrm{Roof}(K)$ and the *floor* $\mathrm{Floor}(K)$ of K are $\mathrm{LRoof}(K) \cup \mathrm{RRoof}(K)$ and $\mathrm{LFloor}(K) \cup \mathrm{RFloor}(K)$, respectively.

(C) For a set X of planar points, $\mathrm{GInt}(X)$ stands for the *geometric* (i.e., topological) *interior* of X. Let h be a (geometric) polygon with endpoints a and b such that $h \setminus \{a,b\} \subseteq \mathrm{TopInt}(\mathrm{FullRect}(L))$, $a \in \mathrm{LBnd}(L)$, and $b \in \mathrm{RBnd}(L)$. Then h cuts $\mathrm{FullRect}(L)$ into an upper half $\uparrow_{\mathrm{g}} h$ and a lower half $\downarrow_{\mathrm{g}} h$; by convention, $h = \uparrow_{\mathrm{g}} h \cap \downarrow_{\mathrm{g}} h$. Note that $\mathrm{Lit}(I) = \uparrow_{\mathrm{g}}\mathrm{Floor}(I) \cap \downarrow_{\mathrm{g}}\mathrm{Roof}(I)$, and similarly for $\mathrm{Lit}(\mathfrak{p})$.

(D) The *body* $\mathrm{Body}(I)$ of I is the geometric region determined by I; if I has only one neon tube, then $\mathrm{Body}(I)$ is a line segment. For example, in Figure 2, $C_2 \in \mathrm{Lamp}(L_2)$ and $\mathrm{Body}(C_2)$ is yellow-filled.

(E) If I is an internal lamp, then the *circumscribed rectangle* $\mathrm{CircR}(I)$ is the region determined by the interval $[x, \mathrm{Peak}(I)]$ where x is the meet of the leftmost lower cover and the rightmost lower cover of $\mathrm{Peak}(I)$. (Equivalently, x is the meet of all lower covers of $\mathrm{Peak}(I)$.)

Since the edges occurring in Definition 3.3 are the same as the neon tubes of L, the following lemma in the present setting is not surprising.

Lemma 3.6 (Czédli [7], (2.10)). *For the fixed multifork sequence of L, see Lemma 3.2, the set of internal lamps of L is $\{I_j\colon\ 1 \leqslant j \leqslant k\}$ where, for $j \in \{1, \dots, k\}$, the lamp $(\mathrm{Foot}(I_j), \mathrm{Peak}(I_j))$ comes into existence by the jth multifork extension, $\mathrm{CircR}(I_j)$ in $L = L_k$ is the geometric region determined by H_j in L_{j-1}, and $\mathrm{Foot}(I_j) \in L_j \setminus L_{j-1}$.*

Since the multifork extensions in Lemma 3.2 are performed at *distributive* 4-cells, it follows easily that, using the notations of Lemma 3.6, for any $j \in \{1, \dots, k\}$,

$$(3.6)\qquad \begin{cases} \text{the lower covers of } \mathrm{Peak}(I_j) \text{ are the same in } L_j \text{ as in } L = L_k. \\ \text{In particular, } I_j \text{ has the same neon tubes in } L_j \text{ as in } L. \\ \text{Furthermore, if a neon tube } \mathfrak{n} \text{ comes into existence in } L_j, \\ \text{then } \mathrm{EOT}(\mathfrak{n}), \mathrm{LEOT}(\mathfrak{n}), \text{ and } \mathrm{REOT}(\mathfrak{n}) \text{ are the same in } L_j \text{ as in } L. \end{cases}$$

Definition 3.7. With the notation used in Lemma 3.6, let I_i and I_j be lamps of L. If $i < j$, then we say that I_j is *younger* than I_i and I_i is *older* than I_j. (This concept depends on the multifork sequence, but this sequence is always fixed.)

By an *edge segment* we mean a geometric line segment $\mathfrak{g}$ of positive length with endpoints lying on the same edge $\mathfrak{e}$ of (the fixed $\mathcal{C}_1$-diagram of) L. In this case, we say that $\mathfrak{g}$ is an *edge segment of* $\mathfrak{e}$. Based on the fact that the neon tubes of L are exactly the prime intervals occurring in Definition 3.3, we can recall a part of Czédli [7], Definition 2.9 and extend it as follows.

Definition 3.8. Let I and J be lamps of a slim rectangular lattice L.

(A) Let $(I, J) \in \varrho_{\mathrm{foot}}$ mean that $I \neq J$, $\mathrm{Foot}(I) \in \mathrm{Lit}(J)$, and I is an internal lamp.

(B) Let $(I, J) \in \varrho_{\mathrm{OTfoot}}$ mean that $I \neq J$, I is an internal lamp, and J has a neon tube $\mathfrak{n}$ such that $\mathrm{Foot}(I) \in \mathrm{GInt}(\mathrm{LEOT}(\mathfrak{n}))$ or $\mathrm{Foot}(I) \in \mathrm{GInt}(\mathrm{REOT}(\mathfrak{n}))$.

(C) Let $(I, J) \in \varrho_{\mathrm{OTCR}}$ mean that $I \neq J$, I is an internal lamp, and J has a neon tube $\mathfrak{n}$ such that $\mathrm{CircR}(I) \subseteq \mathrm{LEOT}(\mathfrak{n})$ or $\mathrm{CircR}(I) \subseteq \mathrm{REOT}(\mathfrak{n})$.

(D) Let $(I, J) \in \varrho_{\mathrm{CircR}}$ mean that $I \neq J$, I is an internal lamp, and

$$\mathrm{CircR}(I) \subseteq \mathrm{Lit}(J).$$

(E) Let $\mathrm{Lamp}(L)$ be the set of lamps of L, and let "$\leqslant$" be the reflexive and transitive closure of the relation ϱ_{foot}. The relational structure $(\mathrm{Lamp}(L); \leqslant)$ is also denoted by $\mathrm{Lamp}(L)$.

The congruence generated by a pair (x, y) of elements is denoted by $\mathrm{con}(x, y)$.

Lemma 3.9 (Mostly Czédli [7], Lemma 2.11). *If L is a slim rectangular lattice, then $\varrho_{\mathrm{foot}} = \varrho_{\mathrm{CircR}} = \varrho_{\mathrm{OTfoot}} = \varrho_{\mathrm{OTCR}}$, $\mathrm{Lamp}(L) = (\mathrm{Lamp}(L); \leqslant)$ is a poset, and whenever $I \prec J$ in $\mathrm{Lamp}(L)$, then $(I, J) \in \varrho_{\mathrm{foot}}$. Furthermore, we have that $(\mathrm{Lamp}(L); \leqslant) \cong (\mathrm{J}(\mathrm{Con}\, L); \leqslant)$ and the map*

$$(3.7) \qquad \varphi\colon (\mathrm{Lamp}(L); \leqslant) \to (\mathrm{J}(\mathrm{Con}\, L); \leqslant) \text{ defined by } I \mapsto \mathrm{con}(\mathrm{Foot}(I), \mathrm{Peak}(I))$$

is an order isomorphism.

The advantage of this lemma over its precursor, [7], Lemma 2.11, is that $(I, J) \in \varrho_{\mathrm{foot}}$ is a mild condition, which is easy to verify, while $(I, J) \in \varrho_{\mathrm{OTCR}}$ is a strong condition, which gives more chance to draw conclusion from.

Proof. With the exception of "$\varrho_{\mathrm{foot}} = \varrho_{\mathrm{OTfoot}} = \varrho_{\mathrm{OTCR}}$", the lemma is already known; see Czédli [7], Lemma 2.11. So we need only to show the just-mentioned equalities. Clearly, $\varrho_{\mathrm{OTCR}} \subseteq \varrho_{\mathrm{OTfoot}} \subseteq \varrho_{\mathrm{foot}}$. Assume that $I_i, I_j \in \mathrm{Lamp}(L)$ such that $(I_i, I_j) \in \varrho_{\mathrm{foot}}$. Since $\mathsf{S}_7^{(m_i)}$ is not distributive, it follows from (3.4), and Lemmas 3.2 and 3.6 that I_i is younger than I_j, that is, $i > j$. In particular, I_i is an internal lamp. With $m := m_j$, let $\mathfrak{n}_1, \ldots, \mathfrak{n}_m$ be the neon tubes of I_j. As $i > j$, these neon tubes are present in L_{i-1}, and so are their original territories $\mathrm{OT}(\mathfrak{n}_1), \ldots, \mathrm{OT}(\mathfrak{n}_m)$ as well as their essential original territories; see (3.6). By (3.5) applied to L_{i-1}, these territories are separated by polygons consisting of lattice edges. By planarity, these "separating polygons" cannot cross the 4-cell H_i of L_{i-1}; this 4-cell becomes $\mathrm{CircR}(I_i)$ in L_i and in L. So $\mathrm{CircR}(I_i) \subseteq \mathrm{OT}(\mathfrak{n}_t)$ for some $t \in \{1, \ldots, m\}$. But the 4-cell H_i in question cannot have the same top as I_j since the opposite case would contradict the distributivity of H_i in L_{i-1}. (Alternatively, [10], Lemma 6.2 would also lead to a contradiction.) Hence, $\mathrm{CircR}(I_i) = H_i \subseteq \mathrm{EOT}(\mathfrak{n}_t)$. Since $\mathrm{EOT}(\mathfrak{n}_t)$ is the union of its two connected "components", $\mathrm{LEOT}(\mathfrak{n}_t)$ and $\mathrm{REOT}(\mathfrak{n}_t)$, and these components are in a positive geometric distance from each other (provided none of them is the empty set), the planarity of the diagram yields that $\mathrm{CircR}(I_i) = H_i \subseteq \mathrm{LEOT}(\mathfrak{n}_t)$ or $\mathrm{CircR}(I_i) = H_i \subseteq \mathrm{REOT}(\mathfrak{n}_t)$. Hence, $(I_i, I_j) \in \varrho_{\mathrm{OTCR}}$, implying that $\varrho_{\mathrm{OTCR}} \subseteq \varrho_{\mathrm{foot}}$ and completing the proof of Lemma 3.9. □

Since we work with the $\mathcal{C}_1$-diagram of our slim rectangular lattice L, the illuminated sets $\mathrm{Lit}(I)$ and $\mathrm{Foot}(I)$, and so the relation ϱ_{foot} are perfectly described by the geometric structure

$$\mathrm{Str}(L) := (\mathrm{FullRect}(L), \{(\mathrm{Foot}(I), \mathrm{Peak}(I))\colon I \in \mathrm{Lamp}(L)\}). \tag{3.8}$$

$$\begin{cases} \text{In particular, if } L \text{ and } L' \text{ are slim rectangular lattices} \\ \text{such that } \mathrm{Str}(L) = \mathrm{Str}(L'), \text{ then } \mathrm{Lamp}(L) \cong \mathrm{Lamp}(L') \\ \text{and so } \mathrm{Con}\, L \cong \mathrm{Con}\, L'. \end{cases} \tag{3.9}$$

4. Auxiliary statements

The following definition is motivated by ϱ_{OTCR}; see Definition 3.8 and Lemma 3.9.

Definition 4.1. For a slim rectangular lattice L and $J \in \mathrm{Lamp}(L)$, let $\mathfrak{p}$ be a neon tube of J. We say that the *original territory of* $\mathfrak{p}$ *is used* if there is a lamp $I \in \mathrm{Lamp}(L)$ such that $I \neq J$ and $\mathrm{CircR}(I) \subseteq \mathrm{LEOT}(\mathfrak{p})$ or $\mathrm{CircR}(I) \subseteq \mathrm{REOT}(\mathfrak{p})$. If I is such, then we say that I *uses* the *original territory* of $\mathfrak{p}$. If there is no such I, then the original territory of $\mathfrak{p}$ is not used.

Remark 4.2. Lemma 3.9 implies that in Definition 4.1, "$I \neq J$" is equivalent to "$I < J$". Furthermore, $I \neq J$ occurs in Definition 4.1 only for emphasis, so it could be omitted; analogous comments would apply to Lemma 4.3 below.

Lemma 4.3. *For* $\mathfrak{p}$ *and* J *as in Definition* 4.1, *the following four conditions are equivalent.*

(a) *The original territory of* $\mathfrak{p}$ *is used, that is, there is a lamp* I *such that* $\mathfrak{p}$ *is not a neon tube of* I *and* $\mathrm{CircR}(I) \subseteq \mathrm{LEOT}(\mathfrak{p})$ *or* $\mathrm{CircR}(I) \subseteq \mathrm{REOT}(\mathfrak{p})$.

(b) *There is a lamp* $I \in \mathrm{Lamp}(L) \setminus \{J\}$ *such that* $\mathrm{Foot}(I)$ *is in* $\mathrm{GInt}(\mathrm{LEOT}(\mathfrak{p}))$ *or it is in* $\mathrm{GInt}(\mathrm{REOT}(\mathfrak{p}))$.

(c) *There is a lamp* $I \in \mathrm{Lamp}(L) \setminus \{J\}$ *such that* $\mathrm{Foot}(I)$ *is in* $\mathrm{EOT}(\mathfrak{p})$.

(d) *There is a precipitous edge segment in* $\mathrm{EOT}(\mathfrak{p})$.

Furthermore, if a lamp I *satisfies one of* (a), (b), *and* (c), *then it satisfies all the three.*

Proof. Since we never change I to another lamp, the last sentence of the lemma will automatically follow when the equivalence of (a), (b), and (c) has been proved.

Since $\mathrm{Foot}(I) \in \mathrm{GInt}(\mathrm{CircR}(I))$, (a) implies (b). By the equality $\mathrm{EOT}(\mathfrak{p}) = \mathrm{LEOT}(\mathfrak{p}) \cup \mathrm{REOT}(\mathfrak{p})$, we obtain that (b) implies (c).

Next, assume that (c) holds. Then $\mathrm{Foot}(I) \in \mathrm{EOT}(\mathfrak{p}) \subseteq \mathrm{Lit}(J)$ and so $(I, J) \in \varrho_{\mathrm{foot}}$. By Lemma 3.9, $(I, J) \in \varrho_{\mathrm{OTCR}}$ and so $\mathrm{Body}(I) \subseteq \mathrm{CircR}(I) \subseteq \mathrm{Lit}(I)$. Thus, $I_t := I$ is younger than $I_k := J$ in the sense of Definition 3.7, that is, $t > k$; indeed, if $I = I_t$ was older than $J = I_k$, then the 4-cell H_k would not be distributive in L_{k-1}.

In L_k, each of $\mathrm{LEOT}(\mathfrak{p})$, $\mathrm{REOT}(\mathfrak{p})$, and $\mathrm{FullRect}(L_k) \setminus \mathrm{EOT}(\mathfrak{p})$ were unions of 4-cells. Some of these 4-cells could have been divided into smaller ones later, but even in L_{t-1}, each of $\mathrm{LEOT}(\mathfrak{p})$, $\mathrm{REOT}(\mathfrak{p})$, and $\mathrm{FullRect}(L_{t-1}) \setminus \mathrm{EOT}(\mathfrak{p})$ were unions of 4-cells. Hence, $H_t \subseteq \mathrm{LEOT}(\mathfrak{p})$, $H_t \subseteq \mathrm{REOT}(\mathfrak{p})$, or H_t is outside $\mathrm{EOT}(\mathfrak{p})$. Since $\mathrm{Foot}(I) = \mathrm{Foot}(I_t) \in \mathrm{GInt}(H_t)$ and $\mathrm{Foot}(I) \in \mathrm{EOT}(\mathfrak{p})$, H_t was not outside $\mathrm{EOT}(\mathfrak{p})$. Hence, $\mathrm{CircR}(I) = \mathrm{CircR}(I_t) = H_t \subseteq \mathrm{LEOT}(\mathfrak{p})$ or $\mathrm{CircR}(I) \subseteq \mathrm{REOT}(\mathfrak{p})$, whereby the original territory of $\mathfrak{p}$ is used. Thus, (c) implies (a), and we have proved that (a), (b), and (c) are equivalent conditions.

By Remark 4.2, the implication (a) $\Rightarrow$ (d) is trivial.

Finally, assume that (d) holds. Then we have a precipitous edge segment in $\mathrm{LEOT}(\mathfrak{p})$ or in $\mathrm{REOT}(\mathfrak{p})$, say, in $\mathrm{LEOT}(\mathfrak{p})$. By the second half of (3.5), we can assume that a precipitous edge segment lies in $\mathrm{GInt}(\mathrm{LEOT}(\mathfrak{p}))$. This edge segment lies on a neon tube $\mathfrak{q}$ of a lamp I. By planarity and (3.5), $\mathfrak{q}$ cannot cross the four sides bordering (the geometric rectangle) $\mathrm{LEOT}(\mathfrak{p})$, so $\mathfrak{q}$ lies fully in $\mathrm{LEOT}(\mathfrak{p})$. In particular, $\mathrm{Peak}(I) = \mathrm{Peak}(\mathfrak{q}) \in \mathrm{LEOT}(\mathfrak{p})$ and $\mathrm{Foot}(\mathfrak{q}) \in \mathrm{LEOT}(\mathfrak{p})$. Observe that $\mathrm{Peak}(I)$ cannot lie on the lower boundary of $\mathrm{LEOT}(\mathfrak{p})$ since otherwise $\mathfrak{q}$, going down from $\mathrm{Peak}(I)$ with a precipitous slope, could not include an edge segment lying in $\mathrm{LEOT}(\mathfrak{p})$.

Next, let $\mathfrak{r}$ be an arbitrary neon tube of I. It goes down from $\mathrm{Peak}(\mathfrak{r}) = \mathrm{Peak}(I)$ with a precipitous slope. Thus, since $\mathrm{Peak}(\mathfrak{r})$ is not on the lower boundary, (3.5) yields that an edge segment lying on $\mathfrak{r}$ lies also in $\mathrm{GInt}(\mathrm{LEOT}(\mathfrak{p}))$. So $\mathfrak{r}$ satisfies the same condition as $\mathfrak{q}$ above, and it follows that $\mathrm{Foot}(\mathfrak{r}) \in \mathrm{LEOT}(\mathfrak{p})$.

Now let $\mathfrak{r}'$ and $\mathfrak{r}''$ be the leftmost neon tube and the rightmost neon tube of I. If $\mathfrak{r}' = \mathfrak{r}''$, then $\mathfrak{q}$ is the only neon tube of I, and the required $\mathrm{Foot}(I) \in \mathrm{LEOT}(\mathfrak{p})$ follows from $\mathrm{Foot}(I) = \mathrm{Foot}(\mathfrak{q}) \in \mathrm{LEOT}(\mathfrak{p})$. So we can assume that $\mathfrak{r}' \neq \mathfrak{r}''$. Then $\mathrm{Foot}(\mathfrak{r}')$ and $\mathrm{Foot}(\mathfrak{r}'')$, as distinct lower covers of $\mathrm{Peak}(I)$, are incomparable; see (3.6). By a result of Czédli (see [8]) and $\mathrm{Foot}(I) = \mathrm{Foot}(\mathfrak{r}') \wedge \mathrm{Foot}(\mathfrak{r}'')$, the interval $[\mathrm{Foot}(I), \mathrm{Foot}(\mathfrak{r}')]$ is a chain (and so a line segment) of slope $(1, -1)$ while $[\mathrm{Foot}(I), \mathrm{Foot}(\mathfrak{r}'')]$ is a line segment of slope $(1, 1)$. The top endpoints $\mathrm{Foot}(\mathfrak{r}')$ and $\mathrm{Foot}(\mathfrak{r}'')$ of these line segments are in $\mathrm{LEOT}(\mathfrak{p})$, whereby so is their common bottom $\mathrm{Foot}(I)$ by the second half of (3.5). Hence, $\mathrm{Foot}(I) \in \mathrm{LEOT}(\mathfrak{p})$, that is, (a) holds. This completes the proof of Lemma 4.3. □

Let $\mathfrak{p}$ be an internal neon tube of a slim rectangular lattice L. As in Czédli and Schmidt, see [17] (but with different terminology), the *fork determined by* $\mathfrak{p}$ is

$$
(4.1) \qquad \begin{cases} F(\mathfrak{p}) := [\mathrm{lsupp}(\mathrm{Foot}(\mathfrak{p})), \mathrm{Foot}(\mathfrak{p})] \cup [\mathrm{rsupp}(\mathrm{Foot}(\mathfrak{p})), \mathrm{Foot}(\mathfrak{p})] \\ \text{together with the edges of these two intervals and the edge } \mathfrak{p}. \end{cases}
$$

For the particular case when $\downarrow_{L'} \mathrm{Peak}(\mathfrak{p})$ is distributive, the following lemma occurs implicitly in [17].

Lemma 4.4. *If $\mathfrak{p}$ is a neon tube of a slim rectangular lattice L and $L' := L \setminus F(\mathfrak{p})$, see* (4.1), *then L' is meet-subsemilattice of L.*

P r o o f. First, we prove that

$$[\mathrm{lsupp}(\mathrm{Foot}(\mathfrak{p})), \mathrm{Foot}(\mathfrak{p})] = \{x \in L\colon\ \mathrm{lsupp}(x) = \mathrm{lsupp}(\mathrm{Foot}(\mathfrak{p}))\}. \tag{4.2}$$

Denote $\mathrm{Foot}(\mathfrak{p})$ by w and $\mathrm{lsupp}(\mathrm{Foot}(\mathfrak{p}))$ by u; so $u = \mathrm{lsupp}(w)$ and we need to show that $[u, w] = \{x \in L\colon\ \mathrm{lsupp}(x) = u\}$. For $y \in [u, w]$, we have that $u = \mathrm{lsupp}(u) \leqslant \mathrm{lsupp}(y) \leqslant \mathrm{lsupp}(w) = u$. Hence, $y \in \{x \in L\colon\ \mathrm{lsupp}(x) = u\}$ and we obtain that $[u, w] \subseteq \{x \in L\colon\ \mathrm{lsupp}(x) = u\}$. To exclude that "$\subset$" holds here, suppose for contradiction that there is $z \in \{x \in L\colon\ \mathrm{lsupp}(x) = u\}$ such that $z \notin [u, w]$. Then $z = \mathrm{lsupp}(z) \vee \mathrm{rsupp}(z) = u \vee \mathrm{rsupp}(z)$ implies that $u \leqslant z$, and if $\mathrm{rsupp}(z) \leqslant \mathrm{rsupp}(w)$, then $z \leqslant u \vee \mathrm{rsupp}(w) \leqslant w$ would contradict that $z \notin [u, w]$. But $\mathrm{rsupp}(z)$ and $\mathrm{rsupp}(w)$ belong to the same chain, $\mathrm{RBnd}(L)$, so they are comparable, and we obtain that $\mathrm{rsupp}(w) < \mathrm{rsupp}(z)$. Hence, $w = \mathrm{lsupp}(w) \vee \mathrm{rsupp}(w) = u \vee \mathrm{rsupp}(w) \leqslant u \vee \mathrm{rsupp}(z) = \mathrm{lsupp}(z) \vee \mathrm{rsupp}(z) = z$. Now the inequality $w \leqslant z$ and $z \notin [u, w]$ imply that $\mathrm{Foot}(\mathfrak{p}) = w < z$. Taking the meet-irreducibility of $\mathrm{Foot}(\mathfrak{p})$ into account, we have that $\mathrm{Peak}(\mathfrak{p}) \leqslant z$. Thus, $\mathrm{lsupp}(\mathrm{Peak}(\mathfrak{p})) \leqslant \mathrm{lsupp}(z)$. With the notation used in Lemmas 3.2 and 3.6, let I_i be the lamp to which $\mathfrak{p}$ belongs. Then $\mathrm{Peak}(\mathfrak{p}) = \mathrm{Peak}(I_i)$, and it is clear in L_i that $u = \mathrm{lsupp}(\mathrm{Foot}(\mathfrak{p})) < \mathrm{lsupp}(\mathrm{Peak}(I)) = \mathrm{lsupp}(\mathrm{Peak}(\mathfrak{p}))$. Since L_i is a sublattice of L, the inequality $u < \mathrm{lsupp}(\mathrm{Peak}(\mathfrak{p}))$ also holds in L. Combining this with the already established $\mathrm{lsupp}(\mathrm{Peak}(\mathfrak{p})) \leqslant \mathrm{lsupp}(z)$, we obtain that $u < \mathrm{lsupp}(z)$. This contradicts the assumption $z \in \{x \in L\colon\ \mathrm{lsupp}(x) = u\}$ and proves (4.2).

Next, for the sake of contradiction, suppose that L' is not meet-closed. Pick elements $s, c, d \in L$ such that $s = c \wedge d$, $s \in F(\mathfrak{p}) = L \setminus L'$ but $c, d \notin F(\mathfrak{p})$. By (4.1), (4.2), and symmetry, we can assume that $\mathrm{lsupp}(s) = \mathrm{lsupp}(\mathrm{Foot}(p))$. Since the function $L \to \mathrm{LBnd}(L)$ defined by $t \mapsto \mathrm{lsupp}(t)$ is clearly an idempotent meet-endomorphism by (3.1), $\mathrm{lsupp}(s) = \mathrm{lsupp}(c) \wedge \mathrm{lsupp}(d)$. As $\mathrm{LBnd}(L)$ is a chain, $\mathrm{lsupp}(s) \in \{\mathrm{lsupp}(c), \mathrm{lsupp}(d)\}$. Let, say, $\mathrm{lsupp}(s) = \mathrm{lsupp}(c)$. Then $\mathrm{lsupp}(c) = \mathrm{lsupp}(\mathrm{Foot}(p))$, so (4.1) and (4.2) give that $c \in F(\mathfrak{p})$, a contradiction. $\square$

For $I \in \mathrm{Lamp}(L)$, let $\mathrm{NumTube}(I) = \mathrm{NumTube}_L(I)$ denote the number of neon tubes of I. The total number of neon tubes of L is denoted by $\mathrm{NumTube}_{\mathrm{all}}(L)$, so $\mathrm{NumTube}_{\mathrm{all}}(L) := \sum\limits_{I \in \mathrm{Lamp}(L)} \mathrm{NumTube}(I)$.

Lemma 4.5 (Sandwiched Neon Tube Lemma)**.** *For a slim rectangular lattice L, let $\mathfrak{n}_1$, $\mathfrak{p}$, and $\mathfrak{n}_2$ be three consecutive neon tubes of an internal lamp $I \in \mathrm{Lamp}(L)$ such that the original territory of $\mathfrak{p}$ is used but those of $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are not used.*

Then there is a slim rectangular lattice L' such that $\mathrm{Lamp}(L') \cong \mathrm{Lamp}(L)$ *but* $|L'| < |L|$ *and* $\mathrm{NumTube}_{\mathrm{all}}(L') = \mathrm{NumTube}_{\mathrm{all}}(L) - 1$*; in fact, there is an isomorphism* $\varphi\colon \mathrm{Lamp}(L) \to \mathrm{Lamp}(L')$ *such that* $\mathrm{NumTube}(\varphi(I)) = \mathrm{NumTube}(I) - 1$ *and* $\mathrm{NumTube}(\varphi(J)) = \mathrm{NumTube}(J)$ *for all* $J \in \mathrm{Lamp}(L) \setminus \{I\}$*.*

P r o o f. With reference to (4.1), denote by L' the subposet of L that we obtain from L by removing the fork $F(\mathfrak{p})$ determined by $\mathfrak{p}$; see Figure 3 for an illustration. We are going to show that L' does the job. By left-right symmetry, we can assume that $\mathfrak{n}_1$ is to the left of $\mathfrak{p}$ and $\mathfrak{p}$ is to the left of $\mathfrak{n}_2$.

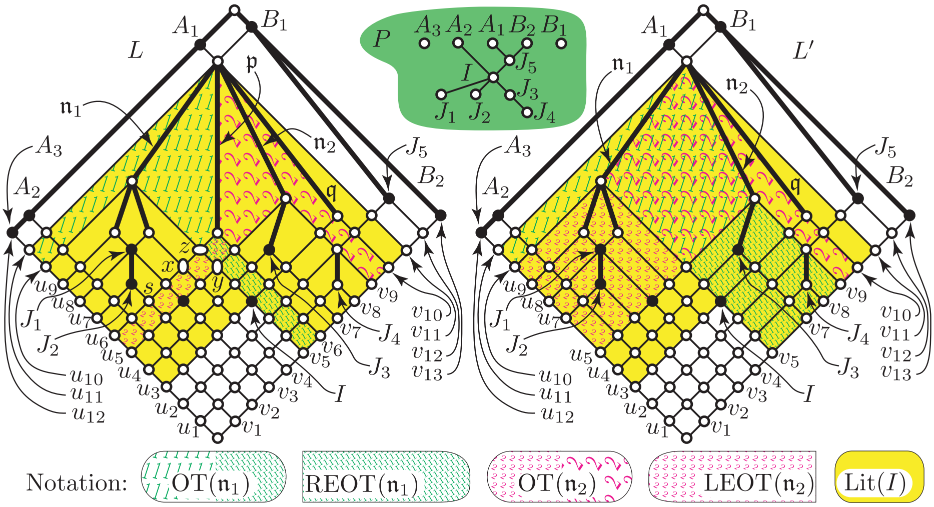

Figure 3. Illustrating the proof of Lemma 4.5 by $\mathrm{Lamp}(L) \cong P \cong \mathrm{Lamp}(L')$.

First, we prove that L' is a sublattice. By a result of Czédli (see [8]),

$$(4.3)\qquad \begin{cases} \text{both intervals occurring in (4.1) are chains of normal slopes.} \\ \text{Hence, by (3.2), } F(\mathfrak{p}) = \mathrm{Floor}(\mathfrak{p}). \end{cases}$$

In Figure 3, these chains are $[u_6, u_6 \vee v_6]$ and $[v_6, u_6 \vee v_6]$. Since none of the original territories of $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are used, we obtain from Lemma 4.3 that

(4.4) none of $\mathrm{REOT}(\mathfrak{n}_1)$ and $\mathrm{LEOT}(\mathfrak{n}_2)$ contains a precipitous line segment.

These two areas border $F(\mathfrak{p}) = \mathrm{Floor}(\mathfrak{p})$ from below. Thus, for any edge $\mathfrak{r}$ of L,

(4.5) if $\mathrm{Peak}(\mathfrak{r}) \in F(\mathfrak{p})$, then $\mathfrak{r}$ is of a normal slope.

For the sake of contradiction, suppose that L' is not join-closed. Then we can pick $x', y' \in L'$ such that $z := x' \vee y' \notin L'$, that is, $z \in F(\mathfrak{p})$. (The join is taken in L.)

By (4.1) and left-right symmetry, we can assume that $z \in [\mathrm{lsupp}(\mathrm{Foot}(\mathfrak{p})), \mathrm{Foot}(\mathfrak{p})]$. In Figure 3, the situation is illustrated with z as the (unique) element drawn by a lying oval. Let $T := [\mathrm{lsupp}(\mathrm{Foot}(\mathfrak{n}_2)), \mathrm{Foot}(\mathfrak{p})]$ in L; it is $[u_5, u_6 \vee v_6]$ in Figure 3. (The area determined by) T is $\mathrm{LEOT}(\mathfrak{n}_2) \subseteq \mathrm{EOT}(\mathfrak{n}_2)$. Hence, by (4.4), T contains no precipitous line segment. Furthermore, as a lattice interval,

(4.6) T is the direct product of a chain and the two-element chain.

Hence, z has only two lower covers, x and y (the standing ovals in the figure), and the edges $[x, z]$ and $[y, z]$ are of normal slopes. Let, say, x be to the left of y. Now $x', y' \in {\downarrow_L} z \setminus \{z\}$, but $\{x', y'\} \nsubseteq {\downarrow_L} y$ since otherwise $z = x' \vee y' \leqslant y \prec z$ would be a contradiction. Hence, at least one of x' and y' is in ${\downarrow_L} z \setminus {\downarrow_L} y \subseteq [\mathrm{lsupp}(\mathrm{Foot}(\mathfrak{p})), \mathrm{Foot}(\mathfrak{p})] \subseteq F(\mathfrak{p})$, contradicting that $x', y' \in L' = L \setminus F(\mathfrak{p})$. Therefore, L' is closed with respect to joins. Since it is also closed with respect to meets by Lemma 4.4, we have proved that L' is a sublattice of L.

Let $\mathfrak{e}$ be an edge in the interval $[\mathrm{lsupp}(\mathrm{Foot}(\mathfrak{p})), \mathrm{Foot}(\mathfrak{p})]$ distinct from the top edge of this interval. Using (4.6), it is clear that if we merge the two 4-cells that share $\mathfrak{e}$ as a common side, we obtain a 4-cell of L'. The situation is similarly for the non-top edges of $[\mathrm{rsupp}(\mathrm{Foot}(\mathfrak{p})), \mathrm{Foot}(\mathfrak{p})]$. The top edges of these two intervals disappear when $\mathrm{Foot}(\mathfrak{p})$ and its two lower covers are omitted, and three "old" 4-cells merge into a "new" 4-cell of L'. Now that we have described the new 4-cells, it follows from (3.3) that L' is a slim rectangular lattice.

It is clear by the paragraph above that with the exception of $\mathfrak{p}$, only some edges of normal slopes are removed when passing from L to L'. The removal of $\mathfrak{p}$ does not influence the pair $(\mathrm{Foot}(I), \mathrm{Peak}(I))$ since $\mathrm{Foot}(I)$ is the meet of the feet of the leftmost neon tube and the rightmost neon tube of I but $\mathfrak{p}$ is a "middle" neon tube of I. Therefore, $\mathrm{Str}(L') = \mathrm{Str}(L)$, see (3.8), and so (3.9) implies that $\mathrm{Lamp}(L') \cong \mathrm{Lamp}(L)$. Finally, since only one neon tube, $\mathfrak{p}$, has been removed, $\mathrm{NumTube}_{\mathrm{all}}(L') = \mathrm{NumTube}_{\mathrm{all}}(L) - 1$. The existence of φ is clear: for $J \in \mathrm{Lamp}(L)$, $\varphi(J)$ is defined by the property $(\mathrm{Foot}(\varphi(J)), \mathrm{Peak}(\varphi(J))) = (\mathrm{Foot}(J), \mathrm{Peak}(J))$. The proof of Lemma 4.5 is complete. □

Lemma 4.6 (No Neighboring Neon Tubes Lemma). *Let L be a slim rectangular lattice. Assume that $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are two neighboring neon tubes of an internal lamp $I \in \mathrm{Lamp}(L)$ such that their original territories are not used. Then there exists a slim rectangular lattice L' such that $|L'| < |L|$ and $(\mathrm{Lamp}(L'); \leqslant) \cong (\mathrm{Lamp}(L); \leqslant)$ but $|\mathrm{NumTube}_{\mathrm{all}}(L')| = |\mathrm{NumTube}_{\mathrm{all}}(L)| - 1$; in fact, there is an order isomorphism $\varphi\colon (\mathrm{Lamp}(L); \leqslant) \to (\mathrm{Lamp}(L'); \leqslant)$ such that $|\mathrm{NumTube}(\varphi(I))| = |\mathrm{NumTube}(I)| - 1$ but $|\mathrm{NumTube}(\varphi(K))| = |\mathrm{NumTube}(K)|$ for any $K \in \mathrm{Lamp}(L) \setminus \{I\}$.*

P r o o f. The proof borrows some ideas from Czédli [10]. Note, however, that the present situation is different from that in [10] since now L', to be defined below, is not a quotient lattice of L in general.

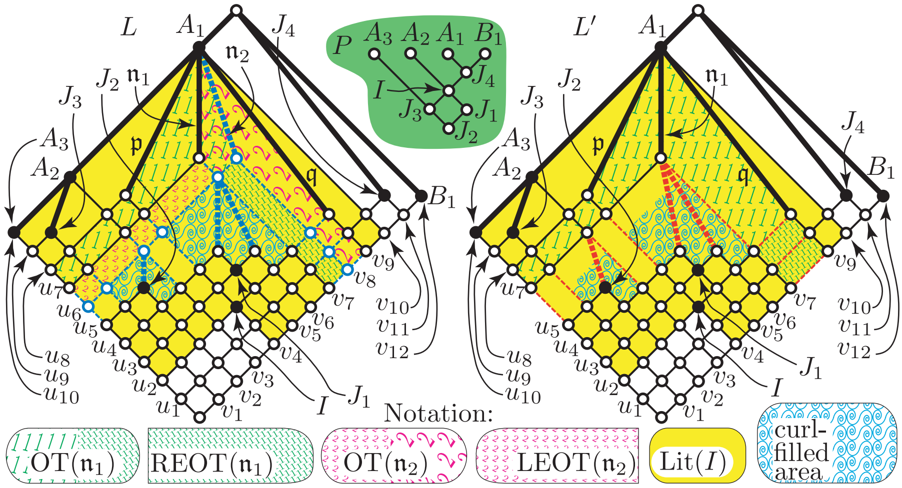

Figure 4. Illustrating the proof of Lemma 4.6 by $\mathrm{Lamp}(L) \cong P \cong \mathrm{Lamp}(L')$.

Let, say, $\mathfrak{n}_2$ be to the right of $\mathfrak{n}_1$; see Figure 4 for an illustration. Observe that, by Lemma 4.3 (or see the figure) and the fact that $\mathrm{REOT}(\mathfrak{n}_1)$ is not used,

$$(4.7)\qquad \begin{cases} \text{the peak of no precipitous edge of } L \text{ belongs to } \mathrm{RFloor}(\mathfrak{n}_2) \text{ and,} \\ \text{in particular, } \mathrm{Foot}(\mathfrak{n}_2) \text{ cannot be the peak of a precipitous edge of } L. \end{cases}$$

Keeping Convention 3.1 in mind, we define L' by describing its $\mathcal{C}_1$-diagram. From (the diagram of) L, we remove the fork $F(\mathfrak{n}_2)$ together with all edges that have one or two endpoints in $F(\mathfrak{n}_2)$. Writing this formally, $L' = L \setminus F(\mathfrak{n}_2)$. On the left of Figure 4, the vertices to be omitted are drawn in blue while the edges to be omitted are the blue dashed edges. Let L' be the set of the remaining vertices (drawn in black). (Note that L' in Figure 4 is not a sublattice of L since $u_4, v_6 \in L'$ but $u_4 \vee_L v_6 \notin L'$.) At this stage, L' with the remaining (black solid) edges is not even a lattice diagram.

Next, let $\mathfrak{q}$ denote the right neighbor of $\mathfrak{n}_2$ among the neon tubes of I or, if $\mathfrak{n}_2$ is the rightmost neon tube of I, then let $\mathfrak{q}$ be the upper right edge of $\mathrm{CircR}(I)$. Actually, it is only $\mathrm{Foot}(\mathfrak{q})$ that we will need, and it is the right neighbor of $\mathrm{Foot}(\mathfrak{n}_2)$ among the lower covers of $\mathrm{Peak}(\mathfrak{n}_2) = \mathrm{Peak}(I)$. For each edge $\mathfrak{r}$ of L, we define or not define an edge $\mathfrak{r}'$ of L' as follows.

$$\text{(4.8)} \quad \begin{cases} \text{If } \mathrm{Foot}(\mathfrak{r}) \in \mathrm{Floor}(\mathfrak{n}_2), \\ \text{then } \mathfrak{r}' \text{ is undefined and } \mathfrak{r} \text{ is called an } \textit{omitted old edge}. \end{cases}$$

$$\text{(4.9)} \quad \begin{cases} \text{If } \mathrm{Foot}(\mathfrak{r}) \notin \mathrm{Floor}(\mathfrak{n}_2) \text{ and } \mathrm{Peak}(\mathfrak{r}) \notin \mathrm{Floor}(\mathfrak{n}_2), \\ \text{then } \mathfrak{r}' := \mathfrak{r} \text{ and } \mathfrak{r} \text{ is called a } \textit{remaining old edge} \text{ of } L'. \end{cases}$$

$$\text{(4.10)} \quad \begin{cases} \text{If } \mathrm{Foot}(\mathfrak{r}) \notin \mathrm{Floor}(\mathfrak{n}_2) \text{ and } \mathrm{Peak}(\mathfrak{r}) \in \mathrm{LFloor}(\mathfrak{n}_2), \\ \text{then let } \mathrm{Foot}(\mathfrak{r}') := \mathrm{Foot}(\mathfrak{r}) \text{ and } \mathrm{Peak}(\mathfrak{r}') := \mathrm{Peak}(\mathfrak{r}) \vee_L \mathrm{lsupp}(\mathrm{Foot}(\mathfrak{n}_1)). \end{cases}$$

$$\text{(4.11)} \quad \begin{cases} \text{If } \mathrm{Foot}(\mathfrak{r}) \notin \mathrm{Floor}(\mathfrak{n}_2) \text{ and } \mathrm{Peak}(\mathfrak{r}) \in \mathrm{RFloor}(\mathfrak{n}_2), \\ \text{then let } \mathrm{Foot}(\mathfrak{r}') := \mathrm{Foot}(\mathfrak{r}) \text{ and } \mathrm{Peak}(\mathfrak{r}') := \mathrm{Peak}(\mathfrak{r}) \vee_L \mathrm{rsupp}(\mathrm{Foot}(\mathfrak{q})). \end{cases}$$

If $\mathfrak{r}$ is in the scope of (4.10) or (4.11), then $\mathfrak{r}'$ and $\mathfrak{r}$ are called a *new edge* and a *changing old edge*, respectively. In Figure 4, $\mathrm{lsupp}(\mathrm{Foot}(\mathfrak{n}_1)) = u_7$, $\mathrm{rsupp}(\mathfrak{q}) = v_9$, and the new edges are the red dashed ones. It follows from (4.7) that each edge $\mathfrak{r}$ of L belongs to the scope of exactly one of (4.8)–(4.11). With its new edges and the remaining old ones, L' turns into a Hasse diagram of a poset $L' = (L; \leqslant)$, which is a subposet of $L = (L; \leqslant)$. Actually, we need to verify that the diagram is a poset diagram. We need to show that no two edges of the new diagram overlap; this will be done a bit later. We also need to show that for every edge $[x, y]$ of the new diagram L', there are no edges $[x, z_1]$, $[z_1, z_2]$, …, $[z_{k-1}, y]$ of L' for some $k \geqslant 2$. This is clear if $[x, y]$ is a new edge, as the only possible $z_1 \in L$ is not in L'; the case when $[x, y]$ is a remaining old edge is even more obvious. To exclude overlapping edges and to show that the poset L' is actually (the diagram of) a slim rectangular lattice, we have to work more. Since none of the original territories $\mathrm{OT}(\mathfrak{n}_1)$ and $\mathrm{OT}(\mathfrak{n}_2)$ is used, Lemmas 3.2 and 3.6 imply the following.

$$\text{(4.12)} \quad \begin{cases} \text{Let } i \in \{1, 2\}. \text{ Then every edge } \mathfrak{r} \text{ in } \mathrm{LEOT}(\mathfrak{n}_i) \text{ is either} \\ \text{of (normal) slope } (1, 1) \text{ and lies on the boundary of } \mathrm{LEOT}(\mathfrak{n}_i) \\ \text{or } \mathfrak{r} \text{ is of (normal) slope } (1, -1). \\ \text{Similarly, every edge } \mathfrak{r} \text{ in } \mathrm{REOT}(\mathfrak{n}_i) \text{ is either of (normal) slope } (1, -1) \\ \text{and lies on the boundary of } \mathrm{REOT}(\mathfrak{n}_i) \text{ or } \mathfrak{r} \text{ is of (normal) slope } (1, 1). \end{cases}$$

Hence, even though L can be more complicated in general than in Figure 4, the original territories indicated by appropriate fill patterns in the figure reflect the general case well. The new edges of L', which originate from changing old edges of L, belong to three categories, which will be discussed separately.

Category 1: We assume that $\mathfrak{r}$ is a precipitous edge in the scope of (4.10). Then $\mathfrak{r}$ is a neon tube of a lamp $J \in \mathrm{Lamp}(L)$ such that $\mathrm{Peak}(J) = \mathrm{Peak}(\mathfrak{r})$ lies on $\mathrm{LFloor}(\mathfrak{n}_2)$. In Figure 4, J can be J_1 or J_2. It follows from (4.12) that we obtain $\mathfrak{r}'$ from $\mathfrak{r}$ by moving the peak of $\mathfrak{r}$ to the northwest along an edge of slope $(1,-1)$. Thus, using that $\mathfrak{r}$ is precipitous, it follows trivially that $\mathfrak{r}'$ is also precipitous; for more details, the reader can (but need not) see (6.8) in [10]. Since no precipitous edge will occur in other categories for changing edges, let us summarize for later references that

$$\text{(4.13)} \qquad \begin{cases} \text{if a precipitous old edge } \mathfrak{h} \text{ of } L \text{ is a changing edge, then it changes to} \\ \text{a precipitous new edge } \mathfrak{h}' \text{ and } \mathrm{Foot}(\mathfrak{h}') = \mathrm{Foot}(\mathfrak{h}). \end{cases}$$

A line or an edge is of a *slight slope* if it is parallel to the vector $(1,t)$ for some $t \in \mathbb{R}$ such that $|t| < 1$. That is, a line or edge is of a slight slope if and only if it is neither of a normal slope nor precipitous. We know from (6.9) of [10] (and it is easy to see) that

$$\text{(4.14)} \qquad \begin{cases} \text{if } l \text{ is a (geometric) line through two distinct lower covers of } \mathrm{Peak}(J), \\ \text{then } l \text{ is of a slight slope.} \end{cases}$$

Next, let $\mathrm{UHCircR}(J)$ stand for the union of the 4-cells whose peaks are $\mathrm{Peak}(J)$; it is a geometric area. (The acronym, taken from [10], comes from "upper half of the circumscribed rectangle".) For $J \in \{J_1, J_2\}$ in Figure 4, $\mathrm{UHCircR}(J)$ in L is curl-filled. Note that on the right of the figure, the curl-filled areas are $\mathrm{UHCircR}(J_1)$ and $\mathrm{UHCircR}(J_2)$ understood in L but not in L'. It follows from Lemmas 3.2 and 3.6 (and, in a different terminology, it is explicitly stated in (6.3) of [10]) that

$$\text{(4.15)} \qquad \begin{cases} \mathrm{GInt}(\mathrm{UHCircR}(J)) \text{ contains no edge segment} \\ \text{that is not a part of a neon tube of } J. \end{cases}$$

Practically, (4.15) means that the curl-filled areas in the figure reflect generality well. Let $\mathfrak{h}'$ be an edge of L' such that $\mathfrak{h}' \neq \mathfrak{r}'$. Since neither the curl-filled area $\mathrm{GInt}(\mathrm{UHCircR}(J))$ nor the 4-cell of $\mathrm{LEOT}(\mathfrak{n}_2)$ that is the upper left neighbor of $\mathrm{CircR}(J)$ contains an edge of L not mentioned in (4.15), $\mathfrak{r}'$ neither crosses nor overlaps $\mathfrak{h}'$ if $\mathfrak{h}$ is of a normal slope. Next, assume that $\mathfrak{h}$ is precipitous and so it is a neon tube and $\mathfrak{h}$ belongs to J, that is, to the same lamp to which $\mathfrak{r}$ belongs. As $\mathrm{Peak}(\mathfrak{h}') = \mathrm{Peak}(\mathfrak{r}')$, the edges $\mathfrak{h}'$ and $\mathfrak{r}'$ do not cross. It follows from (4.14) (applied to the common geometric line that contains both $\mathfrak{h}'$ and $\mathfrak{r}'$) that $\mathfrak{h}'$ and $\mathfrak{r}'$ do not overlap. In the remaining case when $\mathfrak{h}$ is precipitous but not a neon tube of J and $\mathrm{Peak}(\mathfrak{h}) \in \mathrm{LFloor}(\mathfrak{n}_2)$, then let K denote the lamp having $\mathfrak{h}$ as a neon tube. Then K is an internal lamp and $K \neq J$. Since an internal lamp is clearly determined by its peak, $\mathrm{Peak}(J) \neq \mathrm{Peak}(K)$, and they are comparable since $\mathrm{LFloor}(\mathfrak{n}_2)$ where they belong is a chain by (4.3). The role of J and K is interchangeable, so let $\mathrm{Peak}(K) < \mathrm{Peak}(J)$.

Then (the line determined by) RRoof(K) separates J and K, and we obtain easily again that $\mathfrak{r}'$ and $\mathfrak{h}'$ neither cross nor overlap. We have seen that

$$(4.16)\qquad \begin{cases} \text{if } \mathfrak{r}' \text{ originates from a precipitous edge } \mathfrak{r} \text{ of } L, \text{ then } \mathfrak{r}' \text{ neither crosses} \\ \text{nor overlaps any other edge of } L'. \end{cases}$$

Category 2. We assume that $\mathfrak{r}$ is of a normal slope and $\mathfrak{r}'$ is defined in (4.10). Then $b := \mathrm{Peak}(\mathfrak{r}') \in L$ even though $\mathfrak{r}'$ is not an edge of L. It is clear either by Lemmas 3.2 and 3.6 or by comparing the present situation to (4.6) that $\mathrm{Peak}(\mathfrak{r}) \prec_L b$. Hence, $\mathfrak{d} := [\mathrm{Peak}(\mathfrak{r}), b]$ is an edge. This edge lies in $\mathrm{LEOT}(\mathfrak{n}_2)$, and we obtain from (4.12) that $\mathfrak{d}$ is of slope $(1,-1)$. So is $\mathfrak{r}$ since it is of normal slope but does not lie on $\mathrm{LFloor}(\mathrm{Foot}(\mathfrak{n}_2))$. This means that $\mathfrak{r}'$ comes into existence by merging $\mathfrak{r}$ and $\mathfrak{d}$, which are adjacent edges lying on the same line of slope $(1,-1)$. Hence, $\mathfrak{r}'$ is also of slope $(1,-1)$. Therefore, since Category 3 will be analogous to the current one by left-right symmetry and we are armed with (4.13), we can conclude even now that

$$(4.17)\qquad \begin{cases} \text{if } \mathfrak{g} \text{ is a changing old edge of a normal slope, then the edge } \mathfrak{g}' \text{ of } L' \\ \text{is of the same (normal) slope and, furthermore, } \mathfrak{g}' \text{ is obtained} \\ \text{by merging two collinear adjacent edges of } L. \end{cases}$$

It follows from (4.16) and (4.17) that if $\mathfrak{r}'$ crossed or overlapped an edge $\mathfrak{g}'$ of L', then $\mathfrak{g}'$ would be of the other normal slope, $(1,1)$, and it would come into existence by merging $\mathfrak{g}$ to a collinear other edge of L at b. But then $\mathfrak{g}$ would lie on $\mathrm{RFloor}(\mathfrak{n}_2)$ and instead of merging it to a collinear edge to obtain $\mathfrak{g}'$, $\mathfrak{g}$ would have been omitted. Thus,

$$(4.18)\qquad \begin{cases} \text{if } \mathfrak{r} \text{ belongs to Category 2, then } \mathfrak{r}' \text{ neither crosses} \\ \text{nor overlaps any other edge of } L'. \end{cases}$$

Category 3. We assume that $\mathfrak{r}$ is in the scope of (4.11). By (4.7), $\mathfrak{r}$ is of (a normal) slope $(1,1)$. Hence, the situation is basically the left-right symmetric counterpart of the one discussed in Category 2, whereby no details will be given.

Now that the three categories have been investigated, (4.16), (4.18), and the left-right symmetric counterpart of (4.18) for Category 3 imply that L' is a *planar* Hasse diagram. We know from Kelly and Rival (see [23], Corollary 2.4) that planar posets with 0 and 1 are lattices. Hence, L' is a planar lattice. By construction, the number of upper covers of an element $x \in L'$ is the same in L' as in L. Furthermore, an element of L' belongs to the boundary of L' if and only if it belongs to the boundary of L. Therefore, (3.3) and the construction of L' yield in a straightforward but a bit tedious way that L' is a slim rectangular lattice.

Since $x \in L'$ has the same number of covers in L' as in L, we obtain that $\mathrm{M}(L') = L' \cap \mathrm{M}(L)$. Moreover, we already have (4.13) and (4.17), and it is clear that an edge $\mathfrak{r}'$ of L' lies on $\mathrm{Bnd}(L')$ if and only if it lies on $\mathrm{Bnd}(L)$. Clearly, $\mathrm{lc}(L), \mathrm{rc}(L) \in L'$. Therefore, taking the just mentioned facts of the present paragraph and Convention 3.1 (for L) into account, we conclude that L' is (given by) a $\mathcal{C}_1$-diagram.

Since $\mathrm{OT}(\mathfrak{n}_2)$ is not used, it follows from (4.3) and Lemma 4.3 that

$$\text{if } \mathfrak{h} \text{ is a neon tube of } L \text{ and } \mathfrak{h} \neq \mathfrak{n}_2, \text{ then } \mathrm{Foot}(\mathfrak{h}) \notin F(\mathfrak{n}_2) = \mathrm{Floor}(\mathfrak{n}_2). \tag{4.19}$$

It follows from (4.13), (4.17), and the construction of L' that

(4.20)

$$\begin{cases} \text{the neon tubes of } L' \text{ are exactly those } \mathfrak{r}' \text{ where } \mathfrak{r} \text{ is a neon tube of } L \text{ and } \mathfrak{r} \neq \mathfrak{n}_2. \\ \text{Furthermore, for neon tubes } \mathfrak{r} \text{ and } \mathfrak{h} \text{ of } L \text{ such that } \mathfrak{r} \neq \mathfrak{n}_2 \neq \mathfrak{h}, \\ \mathrm{Peak}(\mathfrak{r}') = \mathrm{Peak}(\mathfrak{h}') \text{ if and only if } \mathrm{Peak}(\mathfrak{r}) = \mathrm{Peak}(\mathfrak{h}) \text{ and } \mathrm{Foot}(\mathfrak{r}') = \mathrm{Foot}(\mathfrak{r}). \end{cases}$$

Hence, for a lamp $K \in \mathrm{Lamp}(L) \setminus \{I\}$, $\{\mathfrak{r}' \colon \mathfrak{r} \text{ is a neon tube of } K\}$ is exactly the collection of neon tubes of a lamp K' of L'. Furthermore, $\{\mathfrak{h} \colon \mathfrak{h} \text{ is a neon tube of } I \text{ and } \mathfrak{h} \neq \mathfrak{n}_2\}$ is the set of neon tubes of an internal lamp I' of L'—this is the definition of I'. Note that Lemma 4.4 and (4.20) give that $\mathrm{Foot}(K') = \mathrm{Foot}(K)$ for $K \in \mathrm{Lamp}(L) \setminus \{I\}$. Now (4.20) and the facts mentioned thereafter allow us to conclude that the function $\varphi \colon \mathrm{Lamp}(L) \to \mathrm{Lamp}(L')$ defined by

$$K \mapsto \begin{cases} K' & \text{if } K' \in \mathrm{Lamp}(L') \text{ such that } \mathrm{Foot}(K') = \mathrm{Foot}(K), \\ I' & \text{if } K = I \end{cases} \tag{4.21}$$

is bijective. (Remark that if $\mathfrak{n}_2$ is not the rightmost neon tube of I, then I belongs to the scope of both lines of (4.21).) Note the rule, which follows from (4.20): for any $K \in \mathrm{Lamp}(L)$, we have that $\mathrm{Peak}(\varphi(K)) = \mathrm{Peak}(K)$.

We know from Lemma 3.9 that, in order to see that φ is an order isomorphism, it suffices to show that, for $J, K \in \mathrm{Lamp}(K)$,

$$(J, K) \in \varrho_{\mathrm{foot}} \Leftrightarrow (J', K') \in \varrho_{\mathrm{foot}}. \tag{4.22}$$

Assume that $(J, K) \in \varrho_{\mathrm{foot}}$ and $J \neq I$. Since $\mathrm{Peak}(K')$ is to the northwest (that is, to the $(-1, 1)$ direction) of $\mathrm{Peak}(K)$ or $\mathrm{Peak}(K') = \mathrm{Peak}(K)$, we have that $\mathrm{Lit}(K) \subseteq \mathrm{Lit}(K')$. Hence, $\mathrm{Foot}(J') = \mathrm{Foot}(J) \in \mathrm{Lit}(K) \subseteq \mathrm{Lit}(K')$ gives the required $(J', K') \in \varrho_{\mathrm{foot}}$. If $(I, K) \in \varrho_{\mathrm{foot}}$, then $\mathrm{CircR}(I') = \mathrm{CircR}(I) \subseteq \mathrm{Lit}(K) \subseteq \mathrm{Lit}(K')$ by Lemma 3.9, whereby $(I', K') \in \varrho_{\mathrm{CircR}} = \varrho_{\mathrm{foot}}$, as required. This proves the "$\Rightarrow$" part of (4.22).

Next, assume that $(J', K') \in \boldsymbol{\varrho}_{\text{foot}}$ and $I \notin \{J, K\}$. We know that $\text{Foot}(K') = \text{Foot}(K)$ and $\text{Foot}(J') = \text{Foot}(J)$. If $\text{Peak}(K') = \text{Peak}(K)$, then $\text{Foot}(J) = \text{Foot}(J') \in \text{Lit}(K') = \text{Lit}(K)$ gives the required $(J, K) \in \boldsymbol{\varrho}_{\text{foot}}$. So assume that $\text{Peak}(K') \neq \text{Peak}(K)$. By construction, $\text{Lit}(K') \subseteq \text{Lit}(K) \cup \text{LEOT}(\mathfrak{n}_2)$; see Figure 4. Hence, $\text{Foot}(J) = \text{Foot}(J') \in \text{Lit}(K')$ gives that $\text{Foot}(J) \in \text{Lit}(K)$ or $\text{Foot}(J) \in \text{LEOT}(\mathfrak{n}_2)$. If the second alternative, $\text{Foot}(J) \in \text{LEOT}(\mathfrak{n}_2)$, holds, then $\text{Foot}(J) \subseteq \text{EOT}(\mathfrak{n}_2)$, which contradicts Lemma 4.3 as $\text{OT}(\mathfrak{n}_2)$ is not used. Hence, $\text{Foot}(J) \in \text{Lit}(K)$, which gives that $(J, K) \in \boldsymbol{\varrho}_{\text{foot}}$, as required.

We are left with the case when one of J and K is I.

Assume that $(J', I') \in \boldsymbol{\varrho}_{\text{foot}}$. Then $\text{Foot}(J) = \text{Foot}(J') \in \text{Lit}(I') \subseteq \text{Lit}(I)$ gives the required $(J, I) \in \boldsymbol{\varrho}_{\text{foot}}$. (Note that $\text{Lit}(I') \subset \text{Lit}(I)$ if $\mathfrak{n}_2$ is the rightmost neon tube of I, and $\text{Lit}(I') = \text{Lit}(I)$ otherwise.)

Finally, assume that $(I', K') \in \boldsymbol{\varrho}_{\text{foot}}$. Then $(I', K') \in \boldsymbol{\varrho}_{\text{CircR}}$ by Lemma 3.9. This fact and $\text{CircR}(I) = \text{CircR}(I')$ give that

$$\text{Peak}(I) = \text{Peak}(\text{CircR}(I)) = \text{Peak}(\text{CircR}(I')) \in \text{CircR}(I') \subseteq \text{Lit}(K').$$

Hence, $(\text{Foot}(K'), \text{Peak}(K')) = (\text{Foot}(K), \text{Peak}(K))$, and so $\text{Lit}(K') = \text{Lit}(K)$. These facts lead to $\text{CircR}(I) = \text{CircR}(I') \subseteq \text{Lit}(K') = \text{Lit}(K)$. Thus, $(I, K) \in \boldsymbol{\varrho}_{\text{CircR}} = \boldsymbol{\varrho}_{\text{foot}}$, as required. The proof of Lemma 4.6 is complete. ☐

5. An estimate

The *length* of a lattice K is denoted by $\text{len}(K)$. Our goal is to prove the following statement.

Theorem 5.1. *Let D be a ConSPS-representable distributive lattice with $n := |\mathrm{J}(D)|$ join-irreducible elements. If $n \in \{0, 1\}$, then D is the $(n+1)$-element chain and $K \cong D$. If $n = 2$, then D is the four-element boolean lattice and either $K \cong D$ or K is the three-element chain. If $n \geqslant 3$, then the following two assertions hold.*

(A) *There is a slim rectangular lattice L such that $\text{Con}\, L \cong D$ and*

$$\text{len}(L) \leqslant 2n^2 - 10n + 15, \quad \textit{and so} \quad \text{len}(L) < 2n^2. \tag{5.1}$$

(B) *For any slim semimodular lattice L', if $\text{Con}\, L' \cong D$, then $\text{len}(L') \geqslant n$.*

P r o o f. The case $n \leqslant 2$ is trivial. In the rest of the proof, let $n \geqslant 3$. Let L be a slim rectangular lattice. A trivial induction by Lemmas 3.2 and 3.6 shows that

$$\operatorname{len}(L) = \operatorname{NumTube}_{\mathrm{all}}(L) = |\mathrm{M}(L)|. \tag{5.2}$$

Now if $\operatorname{Con} L \cong D$, then $\operatorname{Lamp}(L) \cong \mathrm{J}(D)$ by Lemma 3.9, so (5.2) gives that $\operatorname{len}(L) = \sum_{I\in\operatorname{Lamp}(L)} \operatorname{NumTube}(I) \geqslant \sum_{I\in\operatorname{Lamp}(L)} 1 = |\operatorname{Lamp}(L)| = n$. Hence, Part (B) holds for the particular case of rectangular SPS lattices.

We know from Grätzer and Knapp (see [21], Theorem 7) and its proof that

$$\begin{cases} \text{each slim semimodular lattice } L' \text{ with at least three elements is a sublattice} \\ \text{of a slim rectangular lattice } L \text{ such that } \operatorname{Con} L \cong \operatorname{Con} L' \text{ and } \operatorname{len}(L) = \operatorname{len}(L'). \end{cases} \tag{5.3}$$

This statement also follows from Lemma 21 in [17] (applied in the reverse directions) and (Corner) Lemma 5.4 proved in [3]. Therefore, Part (B) follows from its particular case mentioned above.

Next, we turn our attention to Part (A). We can assume that $\mathrm{J}(D)$ is not an antichain since otherwise with any grid G of length n and $L := G$, we have that $\operatorname{Con} G \cong D$ and $\operatorname{len}(G) = n \leqslant 2n^2$. Take a slim rectangular lattice L of minimal length such that $\operatorname{Con} L \cong D$. We know from Lemma 3.9 that $\operatorname{Lamp}(L) \cong \mathrm{J}(D)$, and so $|\operatorname{Lamp}(L)| = n$. Let $J \in \operatorname{Lamp}(L)$ be an internal lamp. Let t_J^+ denote the number of neon tubes of J whose original territories are used. Similarly, t_J^- stands for the number of neon tubes of J whose original territories are not used; note that $t_J^+ + t_J^- = \operatorname{NumTube}(J)$. Listing the neon tubes from left to right, let us write a letter u for a used neon tube and a zero for an unused neon tube. Then we obtain a sequence $\vec{s}$ of length $\operatorname{NumTube}(J)$ consisting of t_J^+ u's and t_J^- zeros. Subsequences $0\,u\,0$ and $0\,0$ are forbidden by (5.2) and Lemmas 4.5 and 4.6 since $\operatorname{len}(L)$ is minimal. For another look at $\vec{s}$, take the sequence $\vec{w} := \star u \star u \star u \ldots \star u \star u \star u \star$ of t_J^+ u's and $t_J^+ + 1$ stars that alternate. We can obtain $\vec{s}$ from $\vec{w}$ by removing some stars and replacing the remaining stars by zeros. Observe that only one zero can replace a star since $0\,0$ is a forbidden subsequence. Furthermore, for any two consecutive stars (which occur in a subsequence $\star u \star$), at most one of the two stars can change to 0 and so the other one should be removed since $0\,u\,0$ cannot be a subsequence. Hence, at most every second star can turn to 0 and the rest of the stars are removed. Therefore, the number t_J^- of zeros is at most[4] $\lceil \frac{1}{2}(t_J^+ + 1)\rceil$, the upper integer part of $\frac{1}{2}(t_J^+ + 1)$. Since $\lceil \frac{1}{2}(t_J^+ + 1)\rceil \leqslant t_J^+$, we obtain that, for any $J \in \operatorname{Lamp}(L)$,

$$\operatorname{NumTube}(J) = t_J^+ + t_J^- \leqslant 2 \cdot t_J^+. \tag{5.4}$$

[4] Provided that $t_J^+ > 0$; this correction will be taken into account about seven lines after (5.5).

Let m denote the number of boundary lamps, that is, the number of maximal elements of $\mathrm{Lamp}(L)$ (or, equivalently, those of $\mathrm{J}(D)$). Each of $\mathrm{LBnd}(L)$ and $\mathrm{RBnd}(L)$ contains at least one boundary lamp, whence $m \geqslant 2$. Since $\mathrm{Lamp}(L) \cong \mathrm{J}(D)$ is not an antichain, $m < n$. So $k := n - m$, the number of internal lamps of L, is at least 1. If $\mathfrak{p}$ is a neon tube of an internal lamp J and I uses the original territory of J, then $I < J$ and, in particular, I is also an internal lamp. Furthermore, if $\mathfrak{p}_1, \dots, \mathfrak{p}_{t_J^+}$ denote the neon tubes of J whose original territories are used, then the $\mathrm{GInt}(\mathrm{LEOT}(\mathfrak{p}_1)), \dots, \mathrm{GInt}(\mathrm{LEOT}(\mathfrak{p}_{t_J^+}))$ are pairwise disjoint, and so are $\mathrm{GInt}(\mathrm{REOT}(\mathfrak{p}_1)), \dots, \mathrm{GInt}(\mathrm{REOT}(\mathfrak{p}_{t_J^+}))$. Therefore, using Lemma 4.3 (b), it follows that the lamp I can use the original territories of at most two of the neon tubes of J. The number of lamps I that use the original territory of a neon tube of J is at most $|{\downarrow}J \setminus \{J\}|$, whereby J has at most $2 \cdot |{\downarrow}J \setminus \{J\}|$ neon tubes[5] whose original territories are used. By (5.4), it has at most twice as many neon tubes all together. Hence, the total number of neon tubes of the internal lamps is at most[6]

$$\sum_{\text{internal } J \in \mathrm{Lamp}(L)} 2 \cdot 2 \cdot |{\downarrow}J \setminus \{J\}| = 4 \cdot \sum_{\text{internal } J \in \mathrm{Lamp}(L)} |{\downarrow}J \setminus \{J\}|. \tag{5.5}$$

Observe that $|{\downarrow}J \setminus \{J\}|$ is the number of pairs (I, I') of internal lamps subject to $I < I'$ and $I' = J$. Therefore, the second sum in (5.5) is the number of pairs (I, J) of internal lamps such that $I < J$. This sum reaches its maximum when the internal lamps form a chain. Then there are $\binom{k}{2} = \frac{1}{2}k(k-1)$ such pairs, and so the maximum that (5.5) can take is $2k(k-1)$; it might seem to be an upper bound on the number $\mathrm{NumTube}_{\mathrm{internal}}(L)$ of the neon tubes of the internal lamps of L.

There are two imperfections with the argument above. First, any two minimal internal lamps are incomparable. Hence, letting s denote the number of minimal internal lamps, $\binom{k}{2} = \frac{1}{2}k(k-1)$ has to be reduced by $\binom{s}{2} = \frac{1}{2}s(s-1)$. Second, instead of $2 \cdot |{\downarrow}J \setminus \{J\}| = 0$, a minimal lamp J has exactly one neon tube (trivially or by Lemma 4.5), whereby $s \cdot 1 = s$ has to be added. So we obtain that

$$\begin{aligned} \mathrm{NumTube}_{\mathrm{internal}}(L) &\leqslant 4 \cdot (k(k-1)/2 - s(s-1)/2) + s \\ &= 2k^2 - 2k + 3s - 2s^2 \leqslant' 2k^2 - 2k + 1, \end{aligned} \tag{5.6}$$

where "$\leqslant'$" holds since $3s - 2s^2$ is negative for $s \geqslant 2$ and so we substituted 1 for s.

Next, taking the m boundary lamps, $k = n - m$, and (5.6) into account,

$$\begin{aligned} \mathrm{NumTube}_{\mathrm{all}}(L) &= m + \mathrm{NumTube}_{\mathrm{internal}}(L) \leqslant m + 2(n-m)^2 - 2(n-m) + 1 \\ &= 2n^2 - 2n + 1 + 2 \cdot \underbrace{(m^2 - (2n - 3/2)m)}. \end{aligned} \tag{5.7}$$

[5] For minimal lamps, this will be corrected soon.

[6] To be improved soon by taking the minimal internal lamps of L into account.

Let $f(m) = m^2 - (2n - \frac{3}{2})m$ denote the under-braced term. By the elementary theory of quadratic univariate real functions, $f(m)$ decreases in the closed interval $[0, n - \frac{3}{4}]$. This fact and $2 \leqslant m \leqslant n - 1$ imply that the largest value of $f(m)$ is $f(2) = 7 - 4n$. Substituting this value into (5.7), we obtain that

$$\text{NumTube}_{\text{all}}(L) \leqslant 2n^2 - 10n + 15 < 2n^2. \tag{5.8}$$

Finally, (5.2) and (5.8) complete the proof of Theorem 5.1. □

R e m a r k 5.2. The inequality (5.1) is not sharp. Indeed, no matter which 4-element poset $\mathrm{J}(D)$ is, there is a slim rectangular lattice L such that $|\mathrm{J}(\mathrm{Con}\, L)| \cong D$ and $\mathrm{len}(L) \leqslant 5$ while $2n^2 - 10n + 15$ for $n := 4$ is 7. Note that "$\leqslant 5$" is sharp for $n = 4$; to see this, let $\mathrm{J}(D)$ be the 4-element poset with the "Y-shaped diagram".

Corollary 5.3. *For L in Part* (A) *of Theorem* 5.1, $|L| \leqslant (2n^2 - 10n + 15)^2 < 4n^4$.

P r o o f. By (5.3) and Theorem 5.1, it suffices to show that if L is a slim rectangular lattice of length k, then $|L| \leqslant k^2$. By (1.1), there are chains $C, U \subseteq \mathrm{J}(L)$ such that $\mathrm{J}(L) = C \cup U$. Since $0 \notin C$ and, by rectangularity, $1 \notin C$, $|C| \leqslant k - 1$. Similarly, $|U| \leqslant k - 1$. Since any element of $L \setminus \{0\}$ is of the form $c \vee u$ with $c \in C$ and $u \in U$, L has at most $1 + |C| \cdot |U| = 1 + (k-1)^2 \leqslant k^2$ elements, completing the proof. □

6. Odds and ends

Let P be a poset, and let $j \in P$. We define a new poset P' as follows. The base set of P' is $(P \setminus \{j\}) \cup \{j', j''\}$ where $P \cap \{j', j''\} = \emptyset$. The ordering in P' is defined as follows: for $a, b \in P' \setminus \{j', j''\} = P \setminus \{j\}$, $a \leqslant_{P'} b \Leftrightarrow a \leqslant_P b$, $a \leqslant_{P'} j' \Leftrightarrow a \leqslant_{P'} j'' \Leftrightarrow a \leqslant_P j$, $j' \leqslant_{P'} b \Leftrightarrow j'' \leqslant_{P'} b \Leftrightarrow j \leqslant_P b$, and $j'' \prec_{P'} j'$. We say that P' is obtained from P by *doubling* the *element* j of P. For an example, see P and P' in the middle of Figure 5.

Proposition 6.1. *Let P' be a poset obtained from a JConSPS-representable poset P by doubling a non-maximal element $j \in P$. Then P' is also JConSPS-representable. Furthermore, if L is a slim rectangular lattice such that $P \cong \mathrm{J}(\mathrm{Con}\, L)$, then there is a slim rectangular lattice L' such that $P' \cong \mathrm{J}(\mathrm{Con}\, L')$ and $\mathrm{len}(L') = \mathrm{len}(L) + 2$.*

Czédli in [7], Corollary 3.5 shows that if we double a *maximal* element of a JConSPS-representable poset P, then the new poset P' is never JConSPS-representable.

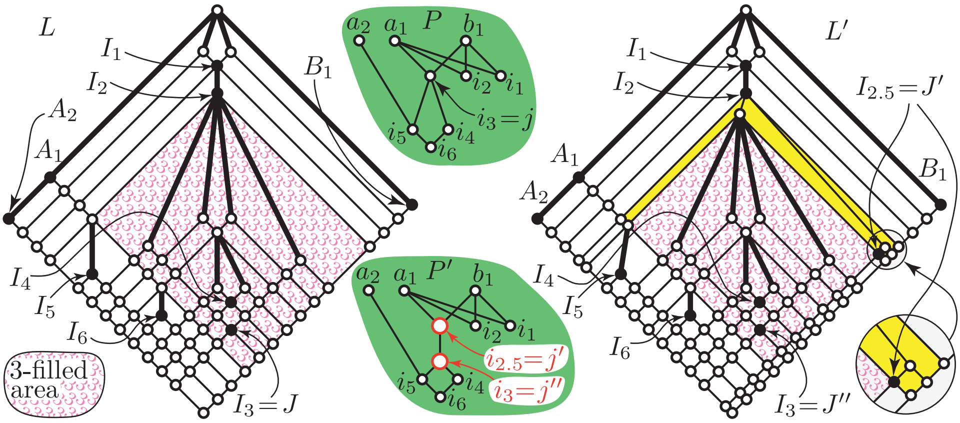

Figure 5. The construction for Proposition 6.1 with a "magnifying glass" at the bottom right.

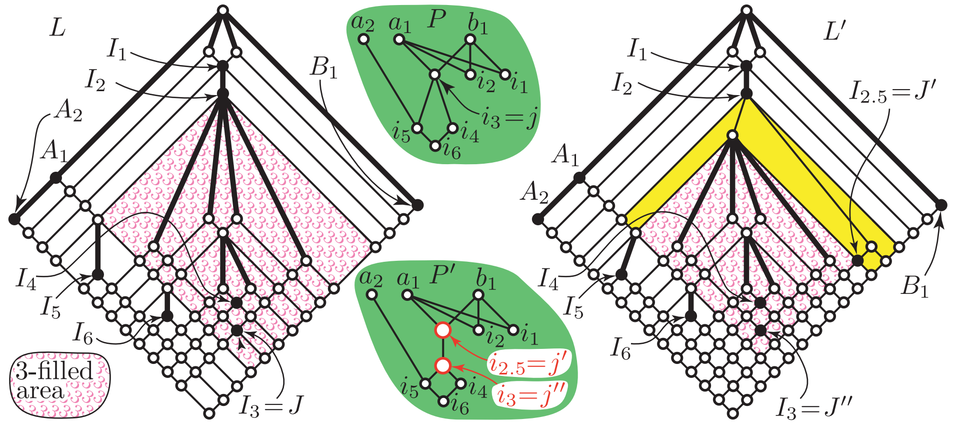

Figure 6. The construction for Proposition 6.1, rescaled.

P r o o f of Proposition 6.1. By Grätzer and Knapp's result, see (5.3), it suffices to deal with the second half of the statement. Assume that L is a rectangular lattice. For $m \in \mathbb{N}^+$, the mth neon tube of a lamp I is understood as the mth neon tube of I from the left; see Convention 3.1. We also count on the fixed multifork sequence of L, see Lemmas 3.2 and 3.6. We know from Lemma 3.9 that there is an order isomorphism $P \to \mathrm{Lamp}(L)$; we denote its action by capitalization, that is, $x \mapsto X$. The notation used in Lemma 3.6 is in effect. Since j is not a maximal element of P, J is an internal lamp; let, say, $J = I_t$. In Figures 5 and[7] 6, $t = 3$. Note that $P \cap P' = P \setminus \{j\} = P \setminus \{j', j''\}$ is a subposet both in P and in P'.

[7] Apart from scaling, the two figures are the same. Figure 5 illustrates the idea of the construction better while Figure 6 is more readable.

For any $x \in P \cap P'$, the lamp corresponding to x is denoted by X both in L and in L'; this should not cause confusion since it will be clear from the context whether $X \in \mathrm{Lamp}(L)$ or $X \in \mathrm{Lamp}(L')$. The pair $(\mathrm{Foot}(X), \mathrm{Peak}(X))$ is the same in L' as in L. So, implicitly, the proof mostly considers lamps as pairs.

We define L' in the following way. Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, be the smallest one out of the geometric lengths of the edges of (the fixed $\mathcal{C}_1$-diagram of) L. With reference to the multifork sequence of L, let $L'_0 := L_0$, $L'_1 := L_1, \dots, L'_{t-1} := L_{t-1}$; these equations also mean the exact coincidence of the corresponding $\mathcal{C}_1$-diagrams in the plane. As for the forthcoming notation, we continue the sequence by $L'_{t-0.5}, L'_t, L'_{t+1}, \dots, L'_k =: L'$. In L'_{t-1} (which is the same as L_{t-1}), let $H'_{t-0.5}$ be the same 4-cell (even geometrically the same) as H_t in L_{t-1}.

Later, H_t turns into $\mathrm{CircR}(I_t)$ in L; in the figure, $\mathrm{CircR}(I_t) = \mathrm{CircR}(I_3)$ is the "3-filled" area in L. In L', only the "major part" of $\mathrm{CircR}(I'_{t-0.5}) = \mathrm{CircR}(I'_{2.5})$ is 3-filled; the rest of $\mathrm{CircR}(I'_{t-0.5}) = \mathrm{CircR}(I'_{2.5})$ is yellow-filled. At H_t in L_{t-1}, we perform a $\mathrm{NumTube}(I_t)$-fold multifork extension, which produces $J = I_t$. (In the figure, where $I_t = I_3 = J$, $\mathrm{NumTube}(I_t) = 4$.) However, in L'_{t-1}, we add a 2-fold multifork at $H'_{t-0.5}$ to obtain a new lattice $L'_{t-0.5}$. Geometrically (in the $\mathcal{C}_1$-diagram), this new multifork extension and the lamp $J' = I_{t-0.5}$ it produces look unusual compared to other figures. Namely, we require that the 4-cell H'_t whose peak is the foot of the leftmost neon tube of J' should be almost as large as $H'_{t-0.5}$. That is, the width η of the "legs" of the Λ-shaped difference $H'_{t-0.5} \setminus H'_t$, which is yellow-filled in the figure, should be very small. (We may think of $\eta = \varepsilon/1000$.) On the right of the figure, $H'_t = H'_3$ in L' is 3-filled.

Next, we perform a $\mathrm{NumTube}(I_t)$-fold multifork extension at H'_t to obtain L'_t from $L_{t-0.5}$ and to produce the lamp $J'' = I_t$ of L'_t (and of L'). The feet of the neon tubes of $J'' = I_t$ in L'_t (and in L') should be the same geometric points as the feet of the neon tubes of $J = I_t$ in L_t (and in L). So the geometric shape of J and that of J'' are almost the same (and they tend to be the same as η tends to 0).

From L'_t, we continue the multifork sequence for L' in the same way as we continue the sequence from L_t to reach L. Even in geometric sense, we do almost the same, that is, with very little differences that would diminish if we formed the limit at $\eta \to 0$. To be more specific, let us agree that we use the alternative notation $I_{-1} = A_1$, $I_{-2} = B_1$, $I_{-3} = A_2$, $I_{-4} = B_2$, $\dots$, $I_{-2k+1} = A_k$, $I_{-2k} = B_k$, $\dots$ for the boundary lamps. (The purpose of this notation is that now each lamp is of the form I_m for some $m \in \mathbb{R}$.) For $s = t, t+1, \dots, k-1$, we select H'_{s+1} as follows. In L_s, the trajectory through the top left edge of the 4-cell H_{s+1} contains exactly one neon tube, $\mathfrak{p}$. Since the top left edge of H_{s+1} is of slope $(1,1)$, it is in the descending part of the trajectory. The neon tube $\mathfrak{p}$ belongs to exactly one lamp, which is older than or as old as I_s; let I_u denote this lamp. Note that we never use the trajectory through the leftmost neon tube of $I_{t-0.5}$ (in the figure, the "narrow" trajectory through the yellow-filled area), whereby $u \neq t - 0.5$ and so u is an integer and I_u will also make sense in L', not only in L.

Among the neon tubes of I_u, let $\mathfrak{p}$ be the αth neon tube (from the left). In L'_t, let $\mathfrak{p}'$ be the αth neon tube of I_u. By left-right symmetry, the top right edge of H_{s+1} defines a neon tube $\mathfrak{q}$ of a lamp I_v in L_s and its counterpart $\mathfrak{q}'$ in L'_s. The top right edge of H_{s+1} is in the ascending part of the trajectory in question. Now we can simply select H'_{s+1} as the unique 4-cell of L'_s where the descending part of the trajectory through $\mathfrak{p}'$ and the ascending part of the trajectory through $\mathfrak{q}'$ cross each other[8]. Once H'_{s+1} has been selected, we perform a $\mathrm{NumTube}(I_{s+1})$-fold multifork extension at this 4-cell of L'_s to obtain L'_{s+1} and its lamp I_{s+1}. This multifork extension should almost be the same geometrically as in the passage from L_s to L_{s+1}; in particular, the feet of the new neon tubes have to be geometrically the same in L'_{s+1} as in L_{s+1}. For later reference, note that

$$\text{(6.1)} \qquad \begin{cases} \text{the left upper edge of } \mathrm{CircR}(I_{s+1}) = H_{s+1} \text{ belongs to the trajectory} \\ \text{through a neon tube of } I_u \text{ both in } L \text{ an } L', \\ \text{and similarly for the right upper edge and } I_v. \end{cases}$$

Finally, we obtain $L' = L'_k$.

Next, in order to recall Czédli [10], Lemma 7.5, we need some notation. Let U be an internal lamp of a slim rectangular lattice K. Then the top edge of the trajectory containing the upper left edge of $\mathrm{CircR}(U)$ is a neon tube of a lamp; we denote this lamp by $\mathrm{Nwl}(U)$. Left-right symmetrically, $\mathrm{Nel}(U)$ stands for the unique lamp that has a neon tube whose trajectory contains the upper right edge of $\mathrm{CircR}(U)$. For a poset Q, let $\mathrm{Min}(Q)$ stand for the set of minimal elements of Q. Now Lemma 7.5 in [10] asserts that if K is a slim rectangular lattice and $U, V \in \mathrm{Lamp}(K)$, then

$$\text{(6.2)} \qquad \begin{cases} U \prec V \text{ in } \mathrm{Lamp}(K) \text{ if and only if } U \text{ is an internal lamp} \\ \text{and } V \in \mathrm{Min}(\{\mathrm{Nwl}(U), \mathrm{Nel}(U)\}). \end{cases}$$

Comparing (6.1) and (6.2) and taking into account that only internal lamps, which all occur in (6.1), can be covered by another lamp, the construction implies that $\mathrm{Lamp}(L)\setminus\{J\}$ is order isomorphic to $\mathrm{Lamp}(L')\setminus\{J', J''\}$. We obtain from Lemma 3.9 that $J' < J''$ in $\mathrm{Lamp}(L')$, $\mathrm{Lamp}(L) \cong \mathrm{Lamp}(L') \setminus \{J'\}$, and $\mathrm{Lamp}(L) \cong \mathrm{Lamp}(L') \setminus \{J''\}$. Thus, using that $P \cong \mathrm{Lamp}(L)$, we conclude that $P' \cong \mathrm{Lamp}(L')$, as required. Furthermore, the construction and (5.2) yield that $\mathrm{len}(L') = \mathrm{len}(L) + 2$.

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[8] The possible doubts whether they cross will be dissolved later.

However, the proof is not complete yet. Indeed, we need to show that the trajectories mentioned earlier do cross in L'_s. To be more precise, we need to show that if the geometric areas $\mathrm{REOT}(\mathfrak{p})$ and $\mathrm{LEOT}(\mathfrak{q})$ cross in L_s, than so do $\mathrm{REOT}(\mathfrak{p}')$ and $\mathrm{LEOT}(\mathfrak{q}')$ in L'_s. Of course, $\mathrm{REOT}(\mathfrak{p}')$ and $\mathrm{LEOT}(\mathfrak{q}')$ are perpendicular if we disregard their thickness but, in principle, they could avoid each other like the right leg of the upper $\wedge$ and the left leg of the lower $\wedge$ do in

$$\wedge\!\!\!\!\!\wedge. \tag{6.3}$$

Fortunately, it is clear by continuity that whenever η is small enough (compared to ε), then $\mathrm{REOT}(\mathfrak{p}')$ and $\mathrm{LEOT}(\mathfrak{q}')$ are close enough to $\mathrm{REOT}(\mathfrak{p})$ and $\mathrm{LEOT}(\mathfrak{q})$, respectively. Thus, since $\mathrm{REOT}(\mathfrak{p})$ and $\mathrm{LEOT}(\mathfrak{q})$ cross each other at a rectangle with sides at least ε, $\mathrm{REOT}(\mathfrak{p}') \cap \mathrm{LEOT}(\mathfrak{q}')$ is a rectangle of a positive area. Furthermore, in L_s, $\mathrm{REOT}(\mathfrak{p}) \cap \mathrm{LEOT}(\mathfrak{q})$ is a 4-cell. Since, except when $J'' = I_t$ was created, $\mathrm{OT}(J') = \mathrm{OT}(I_{t-0.5})$ is never used, we conclude that $\mathrm{REOT}(\mathfrak{p}') \cap \mathrm{LEOT}(\mathfrak{q}')$ is also a 4-cell. This shows that the definition of L'_{s+1} and that of L' make sense, completing the proof of Proposition 6.1. $\square$

Remark 6.2. In most of the cases, the estimate given in (5.1) of Theorem 5.1 is far from being optimal. For example, if $\mathrm{J}(\mathrm{Con}\, L') \cong \mathrm{J}(D) \cong P'$ and P' is obtained from a smaller poset P by doubling a non-maximal element $j \in P$, then, with the notation of Proposition 6.1, the lamp J' corresponding to $j' \in P'$ has only two neon tubes and contributes to $\mathrm{len}(L')$ by 2 regardless the size of $\downarrow_{\mathrm{Lamp}(L')} J'$.

To present another example, let $4 \leqslant n \in \mathbb{N}^+$ and let P_n be the n-element poset consisting of two maximal elements, a and b, $n-3$ minimal elements, $c_1, \ldots, c_{n-3}$, and an element u such that $u \prec a$, $u \prec b$, and $c_i \prec u$ for all $i \in \{1, \ldots, n-3\}$. Then there is a slim rectangular lattice L such that $\mathrm{J}(\mathrm{Con}\, L) \cong P_n$ and $\mathrm{len}(L) = n+1$, which is much smaller than what the estimate (5.1) gives.

In our third example, $3 \leqslant n \in \mathbb{N}^+$ and Q_n is the poset with two maximal elements and $n-2$ minimal elements such that every minimal element is covered by both maximal elements. Then there is a slim rectangular lattice L such that $\mathrm{J}(\mathrm{Con}\, L) \cong Q_n$ and $\mathrm{len}(L) = n$. This example shows that the lower estimate given in Theorem 5.1 (B) cannot be improved.

As Remarks 5.2 and 6.2 allow us to guess, there are many factors that can reduce the number $\mathrm{len}(L) = |\mathrm{NumTube}_{\mathrm{all}}(L)|$ and improve the estimate (5.1). However, it seems to be difficult to take more factors into account without making Theorem 5.1 and the corresponding proof too complicated. Corollary 5.3 is not sharp either. Indeed, in addition to that this corollary is built on the non-sharp Theorem 5.1, there is another reason for this. Namely, if $\mathrm{J}(D) \cong \mathrm{J}(\mathrm{Con}\, L)$ has few non-maximal elements (in particular, if $\mathrm{J}(D)$ is an antichain and so D is boolean), then $|L|$ has few internal lamps and $|L|$ is close to $\mathrm{len}(L)^2$ but then $\mathrm{len}(L)$ is much smaller than what (5.1) gives. On the other hand, if $\mathrm{J}(D)$ has many non-maximal elements, then L has many internal lamps and $|L|$ is considerably smaller than $\mathrm{len}(L)^2$.

Remark 6.3. In order to decide whether a given n-element poset P is JConSPS-representable, it is not economic and usually not even feasible to list all slim rectangular lattices of lengths at most $2n^2 - 10n + 15$; see (2.1) and (5.1), or those of size at most $(2n^2 - 10n + 15)^2$; see Corollary 5.3. It is much faster to rely on the known properties and constructions. To exclude the JConSPS-representability of P in many cases, we can check the known properties of JConSPS-representable posets, see (5.3), Czédli [7], [11], and Czédli and Grätzer [13] (where two earlier properties from Grätzer [18] and [19] are also recalled). To conclude the JConSPS-representability of P and to obtain a slim rectangular lattice L such that $P \cong \mathrm{J}(\mathrm{Con}\, L)$, we can often use the known constructions; see Proposition 6.1, Czédli ([10], Theorems 3.14 and 3.16), and Czédli and Grätzer ([13], Theorem 1.2). If the known properties and constructions do not help, then, compared to what (5.1) gives, the ideas in their proofs radically reduce the number of cases to be inspected for the given P.

If $|P|$ is a small poset, then Remark 6.3 offers a way to decide, in few hours without computers, whether P is JConSPS-representable. (We feel but have not checked that every at most 6-element poset is small in this aspect.) Note that by Czédli [10], Corollary 3.11, each finite poset P that is not JConSPS-representable gives a property (but not always a new property) of JConSPS-representable posets.

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Author's address: *Gábor Czédli*, Department of Algebra and Number Theory, Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, Szeged, Hungary 6720, e-mail: `czedli@math.u-szeged.hu`.