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GENERALIZED DERIVATIONS WITH POWER VALUES ON RINGS AND BANACH ALGEBRAS

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Abstract. Let R be a prime ring and I a nonzero ideal of R. The purpose of this paper is to classify generalized derivations of R satisfying some algebraic identities with power values on I. More precisely, we consider two generalized derivations F and H of R satisfying one of the following identities:

(1) $aF(x)^mH(y)^m = x^ny^n$ for all $x, y \in I$,
(2) $(F(x) \circ H(y))^m = (x \circ y)^n$ for all $x, y \in I$,

for two fixed positive integers $m \geqslant 1$, $n \geqslant 1$ and a an element of the extended centroid of R. Finally, as an application, the same identities are studied locally on nonvoid open subsets of a prime Banach algebra.

Keywords: prime ring; generalized derivation; Banach algebra; Jacobson radical

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1. Introduction

Let R be a ring with center $Z(R)$. Recall that R is a prime if $xRy = 0$ implies $x = 0$ or $y = 0$. For any $x, y \in R$ we write $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product, respectively. An additive mapping $d\colon R \to R$ is a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $F\colon R \to R$ is a *generalized derivation* associated to a derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. A ring R is called primitive if it has a faithful simple module. An ideal P of a ring R is said to be a primitive ideal if P is the annihilator of a simple R-module. The Jacobson radical of a ring R, denoted by $\operatorname{rad}(R)$, is the intersection of all primitive ideals of R. If R has no primitive ideals (i.e., R has no simple modules), then we define $\operatorname{rad}(R) = R$. A Banach algebra is a normed algebra whose underlying

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vector space is a Banach space. The closure of a subset X of a Banach algebra $\mathcal{A}$, denoted by $\overline{X}$, is the intersection of all closed subsets of $\mathcal{A}$ containing X. The interior of a subset X of a Banach algebra $\mathcal{A}$, denoted by $\overset{\circ}{X}$, is the largest open set contained in X. Equivalently, $\overset{\circ}{X}$ is the union of all open subsets of $\mathcal{A}$ contained in X.

During the past few decades, there has been an ongoing interest concerning the relationship between the structure of a prime (semi-prime) ring R and the behavior of generalized derivations of R satisfying some specific algebraic identities on an appropriate subset of R. Motivated by various results in this direction, our aim in this paper is to describe generalized derivations satisfying certain functional identities on a nonzero ideal of a prime ring. Moreover, as an application of our results, we investigate continuous generalized derivations satisfying similar algebraic identities locally on open subsets of a prime Banach algebra.

2. Functional identities on prime rings

The main purpose of this section is to prove the following theorems.

Theorem 2.1. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid, I a nonzero ideal of R, $a \in C$, F and H are generalized derivations of R associated with derivations d and h, respectively, such that*

$$aF(x)^m H(y)^m = x^n y^n \quad \forall\, x, y \in I$$

for two fixed positive integers $m \geqslant 1$ and $n \geqslant 1$. Then $F(x) = \alpha x$, $H(x) = \beta x$, for some $\alpha, \beta \in C$ and $a(\alpha\beta)^m = 1$. Moreover, if $m \neq n$, then $m + n$ is even and $\operatorname{char}(R) = 2^{|m-n|} - 1$.

Theorem 2.2. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid, I a nonzero ideal of R, F and H are generalized derivations of R associated with derivations d and h, respectively, such that*

$$(F(x) \circ H(y))^m = (x \circ y)^n \quad \forall\, x, y \in I$$

for two fixed positive integers $m \geqslant 1$ and $n \geqslant 1$. Then $F(x) = \alpha x$, $H(x) = \beta x$, for some $\alpha, \beta \in C$ and $(\alpha\beta)^m = 1$. Moreover, if $m \neq n$, then $m + n$ is even and $\operatorname{char}(R) = 2^{|m-n|} - 1$.

P r o o f of Theorem 2.1. One can suppose that a, F and H are nonzero, otherwise, the main identity reduces to $x^n y^n = 0$ for all $x, y \in I$. Substituting y by x, we get $x^{2n} = 0$ for all $x \in I$. Using [4], Lemma 1.1, R has a nonzero nilpotent ideal, which contradicts the primeness of R.

Since I and Q_r satisfy the same differential identities (see [8], Theorem 2) we may assume that

$$aF(x)^m H(y)^m = x^n y^n \quad \forall\, x, y \in Q_r. \tag{2.1}$$

Using [9], Theorem 3, there exist $\alpha, \beta \in Q_r$ such that $F(x) = \alpha x + d(x)$ and $H(x) = \beta x + h(x)$. Hence, equation (2.1) becomes

$$a(\alpha x + d(x))^m (\beta y + h(y))^m = x^n y^n \quad \forall\, x, y \in Q_r. \tag{2.2}$$

Case 1: If d and h are both Q_r-inner, then there exist $q_1, q_2 \in Q_r$ such that $d(x) = [q_1, x]$, $h(x) = [q_2, x]$ for all $x \in Q_r$, thus

$$P(x, y) = a(\alpha x + [q_1, x])^m (\beta y + [q_2, y])^m - x^n y^n = 0 \quad \forall\, x, y \in Q_r.$$

In view of [3], Theorem 2.5 and Theorem 3.5, we know that both Q_r and $Q_r \bigotimes_C \overline{C}$ are centrally closed, where $\overline{C}$ is the algebraic closure of C. We may replace Q_r by itself or $Q_r \bigotimes_C \overline{C}$ according whether C is finite or infinite. Therefore we may assume that Q_r is centrally closed over C, which is either finite or algebraically closed. By Martindale's theorem (see [10]), Q_r is a primitive ring having a nonzero socle H with C the associated division ring. In light of Jacobson's theorem (see [5], page 75), Q_r is isomorphic to a dense ring of linear transformations on a vector space V over C.

If $\dim_C V = k$, then the density of Q_r gives $Q_r \cong M_k(C)$.

Assume that $\dim_C V \geqslant 2$. We want to show that $\{u, q_1 u\}$ are linearly C-dependent for all $u \in V$. Indeed, suppose that u and $q_1 u$ are linearly C-independent.

If $q_2 u \notin \mathrm{Span}_C\{u, q_1 u\}$, then $\{u, q_1 u, q_2 u\}$ is C-independent, invoking [2], Definition 5.11. There exist $f, g \in Q_r$ such that $fu = 0$, $fq_1 u = -u$, $fq_2 u = u$, $gu = 0$, $gq_1 u = u$, $gq_2 u = -u$,

$$P(f, g)u = (a(\alpha f + [q_1, f])^m (\beta g + [q_2, g])^m - f^n g^n)u = 0. \tag{2.3}$$

It is obvious that $(\alpha f + [q_1, f])^m u = u$, $(\beta g + [q_2, g])^m u = u$ and $f^n g^n u = 0$. Therefore $P(f, g)u = au = 0$ for all $u \in V$, a contradiction.

Let now $q_2 u \in \mathrm{Span}_C\{u, q_1 u\}$. Then $q_2 u = \lambda u + \mu q_1 u$ for some $\lambda, \mu \in C$, hence $gq_2 u = \lambda gu + \mu g q_1 u = \mu u$, so $(\beta g + [q_2, g])^m u = \mu^m u$, consequently $P(f, g)u = a\mu^m u = 0$ for all $u \in V$, which is absurd.

Then in all cases, $\{u, q_1 u\}$ are linearly C-dependent for all $u \in V$, that is, $q_1 u = \lambda_u u$ for some $\lambda_u \in C$. Obviously, for any $v \in V$ such that $\{u, v\}$ are linearly C-independent, we have $q_1(u - v) = \lambda_u u - \lambda_v v = \lambda_{u-v}(u - v)$, then $(\lambda_u - \lambda_{u-v})u - (\lambda_v - \lambda_{u-v})v = 0$, hence $\lambda_u = \lambda_{u-v} = \lambda_v$, finally $q_1 u = \lambda u$ for all $u \in V$.

On the other hand, for $r \in R$ and $u \in V$ we get

$$(rq_1)u = r(q_1u) = r\lambda u = \lambda(ru) = q_1(ru) = (q_1r)u,$$

then $[R, q_1]V = 0$, thus $q_1 \in C$. Similarly, we prove that $q_2 \in C$. The main equation becomes

$$Q(x, y) = a(\alpha x)^m(\beta y)^m - x^n y^n = 0 \quad \forall\, x, y \in Q_r. \tag{2.4}$$

Now we aim to prove that $\{w, \alpha w\}$ are linearly C-dependent for all $w \in V$, indeed, suppose that w and αw are linearly C-independent.

If $\beta w \notin \mathrm{Span}_C\{w, \alpha w\}$, then $\{w, \alpha w, \beta w\}$ are C-independent, Q_r being a dense ring of linear transformation of V. It follows that there exist $f, g \in Q_r$ such that $fw = 0$, $f\alpha w = w$, $f\beta w = w$, $gw = w$, $g\alpha w = 0$, $g\beta w = w$,

$$Q(f, g)w = (a(\alpha f)^m(\beta g)^m - f^n g^n)w = 0. \tag{2.5}$$

Firstly

$$\begin{aligned}(a(\alpha f)^m(\beta g)^m - f^n g^n)w &= (a(\alpha f)^m(\beta g)^{m-1}(\beta g w) - f^n g^{n-1}(gw))\\ &= (a(\alpha f)^{n-1}(\alpha f)\beta w - f^{n-1}(fw)) = a\alpha w.\end{aligned}$$

Using relation (2.5), we get $Q(f, g)w = a\alpha w = 0$ for all $w \in V$, a contradiction. Now, if $\beta w \in \mathrm{Span}_C\{w, \alpha w\}$, then $\beta w = \lambda_1 w + \lambda_2 \alpha w$ for some $\lambda_1, \lambda_2 \in C$. It follows that $g\beta w = \lambda_1 w$ and $f\beta w = \lambda_2 w$, thus

$$\begin{aligned}Q(f, g)w &= (a(\alpha f)^m(\beta g)^{m-1}(\beta g w) - f^n g^{n-1}(gw)) = (a(\alpha f)^m(\beta g)^{m-2}\beta\lambda_1 w)\\ &= (\lambda_1)^{m-1}(a(\alpha f)^m \beta w) = (\lambda_1)^{m-1}(a(\alpha f)^{m-1}\alpha(f\beta w))\\ &= (\lambda_1)^{m-1}\lambda_2(a(\alpha f)^{m-1}\alpha w) = (\lambda_1)^{m-1}\lambda_2 a\alpha w.\end{aligned}$$

Using relation (2.5), we get $Q(f, g)w = (\lambda_1)^{m-1}\lambda_2 a\alpha w = 0$ for all $w \in V$, which is absurd. Then in all cases, $\{w, \alpha w\}$ are linearly C-dependent for all $w \in V$, thus $\alpha w = \gamma_w w$ for all $w \in V$ and for some $\gamma_w \in C$. It is straightforward that $\alpha w = \gamma w$, thus $[R, \alpha]V = 0$ and $\alpha \in C$. Analogously, we prove that $\beta \in C$. Then the main equation reduces to

$$a(\alpha\beta)^m x^m y^m - x^n y^n = 0 \quad \forall\, x, y \in I. \tag{2.6}$$

If $m = n$, then $a(\alpha\beta)^m = 1$ directly follows.

On the other hand, if $m \neq n$, invoking [7], Lemma 1, $I \subseteq M_s(K)$ for a field K and an integer $s > 1$, then $M_s(K)$ satisfies

$$a(\alpha\beta)^m x^m y^m - x^n y^n = 0. \tag{2.7}$$

Taking e_{ii} instead of x and y in relation (2.7) for a fixed positive integer $i \leqslant s$, we get $(a(\alpha\beta)^m - 1)e_{ii} = 0$, then $a(\alpha\beta)^m = 1$. Now equation (2.7) becomes

$$x^m y^m - x^n y^n = 0 \quad \forall\, x, y \in M_s(K). \tag{2.8}$$

Suppose that $m+n$ is odd, then taking $-y$ instead of y in equation (2.8), we obtain

$$x^m y^m + x^n y^n = 0 \quad \forall\, x, y \in M_s(K). \tag{2.9}$$

Summing relation (2.8) and equation (2.9), we find that $x^m y^m = 0$. In particular, for $x = y = e_{11}$, the last equation yields a contradiction.

Now if $m+n$ is even, taking $2e_{jj}$ instead of x and e_{jj} instead of y in relation (2.8) for a fixed positive integer $j \leqslant s$, we get $2^m e_{jj} - 2^n e_{jj} = 0$, that is $(2^{|m-n|} - 1)e_{jj} = 0$, which is impossible unless $\operatorname{char}(R) = 2^{|m-n|} - 1$.

Case 2: If d and h are linearly C-independent modulo inner derivations of $Q_r(R)$, then using [6], Theorem 2 along with relation (2.2), we get

$$a(\alpha x + z_1)^m (\beta y + z_2)^m = x^n y^n \quad \forall\, x, y, z_1, z_2 \in Q_r. \tag{2.10}$$

In particular, for $x = y = 0$, equation (2.10) reduces to $a z_1^m z_2^m = 0$ for all $z_1, z_2 \in Q_r$, which contradicts [4], Lemma 1.1.

Case 3: If d and h are linearly C-dependent modulo inner derivations of $Q_r(R)$, then we may suppose that $d(x) = \delta h(x) + [q, x]$ for all $x \in R$ with $\delta \in C \setminus \{0\}$ and $q \in Q_r(R)$. Note that h is Q_r-outer, otherwise d and h are both Q_r-inner, which has already been treated before in Case 1. The main equation becomes

$$a(\alpha x + \delta h(x) + [q, x])^m (\beta y + h(y))^m - x^n y^n = 0 \quad \forall\, x, y \in Q_r. \tag{2.11}$$

Theorem 2 of [6] yields

$$a(\alpha x + \delta z_1 + [q, x])^m (\beta y + z_2)^m - x^n y^n = 0 \quad \forall\, x, y, z_1, z_2 \in Q_r.$$

Arguing as in Case 2, we also get a contradiction. □

P r o o f of Theorem 2.2. We may suppose that F and H are nonzero. Indeed, otherwise the main identity becomes $(x \circ y)^n = 0$ for all $x, y \in I$. Taking x instead of y, we get $2^n x^{2n} = 0$ for all $x \in I$. Invoking char $(R) \neq 2$ along with [4], Lemma 1.1, R has a nonzero nilpotent ideal, which contradicts the primeness of R.

Now using [8], Theorem 2, our hypothesis yields

$$(F(x) \circ H(y))^m = (x \circ y)^n \quad \forall\, x, y \in Q_r. \tag{2.12}$$

By view of [9], Theorem 3, there exist $\alpha, \beta \in Q_r$ such that $F(x) = \alpha x + d(x)$ and $H(x) = \beta x + h(x)$, then relation (2.12) yields

$$((\alpha x + d(x)) \circ (\beta y + h(y)))^m = (x \circ y)^n \quad \forall\, x, y \in Q_r.$$

Case 1: d and h are both Q_r-inner, then there exist $q_1, q_2 \in Q_r$ such that $d(x) = [q_1, x]$, $h(x) = [q_2, x]$ for all $x \in Q_r$, hence

$$P(x, y) = ((\alpha x + [q_1, x]) \circ (\beta y + [q_2, y]))^m - (x \circ y)^n = 0 \quad \forall\, x, y \in Q_r.$$

By adopting a similar approach to the one used in Theorem 2.1, it follows that Q_r is isomorphic to a dense ring of linear transformation of vector space V over C.

Assume that $\dim_C V \geqslant 2$, clearly $\{v, q_1 v\}$ are linearly C-dependent for all $v \in V$, otherwise, we suggest to suppose that v and $q_1 v$ are linearly C-independent.

If $q_2 v \notin \mathrm{Span}_C\ \{v, q_1 v\}$, then $\{v, q_1 v, q_2 v\}$ is C-independent. Using the density of Q_r, there exist $f, g \in Q_r$ such that $fv = 0$, $fq_1 v = -v$, $fq_2 u = v$, $gv = 0$, $gq_1 v = -v$, $gq_2 v = v$,

$$P(f, g)v = (((\alpha f + [q_1, f]) \circ (\beta g + [q_2, g]))^m - (f \circ g)^n)v = 0. \tag{2.13}$$

The only nonzero terms are $[q_1, f][q_2, g]v = -v$ and $[q_2, g][q_1, f]v = -v$.

$$\begin{aligned} P(f, g)v &= (([q_1, f][q_2, g] + [q_2, g][q_1, f])^m - (f \circ g)^n)v \\ &= ([q_1, f][q_2, g] + [q_2, g][q_1, f])^m v \\ &= ([q_1, f][q_2, g] + [q_2, g][q_1, f])^{m-1}(-2v) = (-2)^m v. \end{aligned}$$

Invoking equation (2.13), $P(f, g)v = (-2)^m v = 0$ for all $v \in V$, a contradiction. Now if $q_2 v \notin \mathrm{Span}_C\{v, q_1 v\}$, then $q_2 v = \lambda v + \mu q_1 v$ for some $\lambda, \mu \in C$, thus $gq_2 v = \lambda gv + \mu g q_1 v = -\mu v$, accordingly $P(f, g)u = (2\mu)^m v = 0$ for all $w \in V$, which is also impossible.

Then generally $\{v, q_1 v\}$ are linearly C-dependent for all $v \in V$. Simple arguments lead to $q_1 v = \lambda v$ for all $v \in V$ with $\lambda \in C$, which, as in the proof of Theorem 2.1, forces $q_1 \in C$. Using a similar argument, we get $q_2 \in C$.

We propose to prove that $\{w, \alpha w\}$ are linearly C-dependent for any $w \in V$. Suppose that $\{w, \alpha w\}$ are linearly C-independent.

If $\beta w \notin \mathrm{Span}_C\{w, \alpha w\}$, then $\{w, \alpha w, \beta w\}$ are C-independent, Q_r being a dense ring of linear transformation of V, it follows that there exist $f, g \in Q_r$ such that $fw = w$, $f\alpha w = 0$, $f\beta w = w$, $gw = 0$, $g\alpha w = w$, $g\beta w = 0$,

$$Q(f,g)w = (((\alpha f) \circ (\beta g))^m - (f \circ g)^n)w = 0. \tag{2.14}$$

Firstly, $(\alpha f\beta g + \beta g\alpha f)w = \beta g\alpha w = \beta w$ and $(\alpha f\beta g + \beta g\alpha f)\beta w = \beta w$. Consequently, $Q(f,g)w = \beta w = 0$ for all $w \in V$, which is impossible.

Let now $\beta w \in \mathrm{Span}_C\{w, \alpha w\}$, then $\beta w = \mu_1 w + \mu_2\alpha w$ for some $\mu_1, \mu_2 \in C$, accordingly, we get $(\alpha f\beta g + \beta g\alpha f)\beta w = \mu_1(\mu_2\alpha + \beta)w$. It is obvious that

$$(\alpha f\beta g + \beta g\alpha f)\mu_1(\mu_2\alpha + \beta)w = \mu_1(\mu_2\alpha + \beta)w.$$

Then

$$Q(f,g)w = \mu_1(\mu_2\alpha + \beta)w$$

for any $w \in V$, which is impossible. Then $\{w, \alpha w\}$ are linearly C-dependent for any $w \in V$. Simple computations lead to $\alpha w = \gamma w$ for some $\gamma \in C$, thus $[R, \alpha]V = 0$, then $\alpha \in C$. Likewise, we get $\beta \in C$. Returning to the main equation, we find that

$$(\alpha\beta)^m(x \circ y)^m = (x \circ y)^n \quad \forall\, x, y \in I. \tag{2.15}$$

If $m = n$, then $(\alpha\beta)^m = 1$ follows immediately.

Regarding the case where $m \neq n$, in light of [7], Lemma 1, $M_s(K)$ satisfies

$$(\alpha\beta)^m(x \circ y)^m = (x \circ y)^n \tag{2.16}$$

for a field K and an integer $s > 1$. Taking e_{ij} instead of x and e_{ji} instead of y in relation (2.16) for some fixed positive integers $i, j \leqslant s$, we get $((\alpha\beta)^m - 1)(e_{ii} + e_{jj}) = 0$, which implies $(\alpha\beta)^m = 1$. Now relation (2.16) reduces to

$$(x \circ y)^m - (x \circ y)^n = 0 \quad \forall\, x, y \in M_s(K). \tag{2.17}$$

If $m + n$ is odd, then substituting y by $-y$ in equation (2.17), we obtain

$$(x \circ y)^m + (x \circ y)^n = 0 \quad \forall\, x, y \in M_s(K). \tag{2.18}$$

The summation of relation (2.17) and equation (2.18) gives $(x \circ y)^m = 0$ for all $x, y \in M_s(K)$. Setting $x = y = I_s$ with I_s the matrix identity of $M_s(K)$, we get to $2^m I_s = 0$ so that $I_s = 0$, a contradiction.

Now if $m+n$ is even, for $x = y = e_{ii}$, equation (2.17) yields

$$(2e_{ii})^m - (2e_{ii})^n = 0.$$

That is, $(2^{|m-n|} - 1)e_{ii} = 0$, which is impossible unless $\mathrm{char}\,(R) = 2^{|m-n|} - 1$.

Case 2: If d and h are linearly C-independent modulo inner derivations of $Q_r(R)$, then the main identity along with [6], Theorem 2 give

$$((\alpha x + z_1) \circ (\beta y + z_2))^m = (x \circ y)^n \quad \forall\, x, y, z_1, z_2 \in Q_r. \tag{2.19}$$

Setting $x = y = 0$, equation (2.19) reduces to $(z_1 \circ z_2)^m = 0$ for all $z_1, z_2 \in Q_r$; a contradiction follows directly from [4], Lemma 1.1.

Case 3: If d and h are linearly C-dependent modulo inner derivations of $Q_r(R)$, then one can show that $d(x) = \delta h(x) + [q, x]$ for some $\delta \in C \setminus \{0\}$, $q \in Q_r(R)$ and h is necessarily Q_r-outer. The main equation becomes

$$((\alpha x + \delta h(x) + [q, x]) \circ (\beta y + h(y)))^m - (x \circ y)^n = 0 \quad \forall\, x, y \in Q_r. \tag{2.20}$$

Theorem 2 of [6] yields

$$((\alpha x + \delta z_1 + [q, x]) \circ (\beta y + z_2))^m - (x \circ y)^n = 0 \quad \forall\, x, y, z_1, z_2 \in Q_r.$$

An approach similar to that adopted in Case 2 leads to a contradiction. □

3. Application on prime Banach algebras

Throughout this section, $\mathcal{A}$ denotes a real or complex Banach algebra. To prove our main results we need the following lemma.

Lemma 3.1 ([1]). *Let $\mathcal{A}$ be a Banach algebra. If $P(t) = \sum_{k=0}^{n} b_k t^k$ is a polynomial in the real variable t with coefficients in $\mathcal{A}$, and if for an infinite set of real values of t, $P(t) \in M$, where M is a closed linear subspace of $\mathcal{A}$, then every b_k lies in M.*

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra, $Q_{\mathcal{A}}$ its right Martindale quotient ring, $C_{\mathcal{A}}$ its extended centroid, $F = L_\alpha + d$, $H = L_\beta + h$ are two continuous generalized derivations with L_α (or L_β) the left multiplication by an element $\alpha \in \mathcal{A}$ (or $\beta \in \mathcal{A}$), d, h derivations of $\mathcal{A}$, $a \in C_{\mathcal{A}}$, $m \geqslant 1$ and $n \geqslant 1$ are two fixed positive integers such that*

$$aF(x)^m H(y)^m - x^n y^n \in \mathrm{rad}(\mathcal{A}) \quad \forall\, x, y \in \mathcal{A},$$

then $d(\mathcal{A}) \subseteq \mathrm{rad}(\mathcal{A})$ and $h(\mathcal{A}) \subseteq \mathrm{rad}(\mathcal{A})$. Moreover, if $\mathcal{A}$ is primitive, then $F(x) = \alpha x$, $H(x) = \beta x$, for some $\alpha, \beta \in C_{\mathcal{A}}$ with $a(\alpha\beta)^m = 1$ and $m = n$.

P r o o f. Let P be a primitive ideal, set $F_P, H_P\colon \mathcal{A}/P \to \mathcal{A}/P$ with $F_P(\overline{x}) = F_P(x+P) = F(x)+P$ and $H_P(\overline{x}) = H_P(x+P) = H(x)+P$ for all $\overline{x} \in \mathcal{A}/P$. Invoking [11], Theorem 2.2 primitive ideals are invariant under F and H, then F_P and H_P are well defined. P being primitive, Lemma 5.36 of [2] implies that $\mathcal{A}/P$ is a primitive ring and thus prime by [2], Lemma 5.4. The main identity becomes

$$aF_P(x)^m H_P(y)^m - x^n y^n = 0 \quad \forall\, x, y \in \mathcal{A}/P.$$

Using Theorem 2.1, we get $d_P = 0$ and $h_P = 0$, that is, $d(\mathcal{A}) \subseteq P$ and $h(\mathcal{A}) \subseteq P$ for any primitive ideal P. Then $d(\mathcal{A}) \subseteq \operatorname{rad}(\mathcal{A})$ or $h(\mathcal{A}) \subseteq \operatorname{rad}(\mathcal{A})$. Moreover, if $\mathcal{A}$ is primitive, then $\operatorname{rad}(\mathcal{A}) = (0)$. Invoking again Theorem 2.1, we get the required results. □

Using the same arguments as above, with a suitable modification, application of Theorem 2.2 yields the following result.

Theorem 3.2. *Let $\mathcal{A}$ be a Banach algebra, $F = L_\alpha + d$, $H = L_\beta + h$ be two continuous generalized derivations with L_α (or L_β) the left multiplication by an element $\alpha \in \mathcal{A}$ (or $\beta \in \mathcal{A}$), d, h derivations of $\mathcal{A}$, $m \geqslant 1$ and $n \geqslant 1$ be two fixed positive integers such that*

$$(F(x) \circ H(y))^m - (x \circ y)^n \in \operatorname{rad}(\mathcal{A}) \quad \forall\, x, y \in \mathcal{A},$$

then $d(\mathcal{A}) \subseteq \operatorname{rad}(\mathcal{A})$ and $h(\mathcal{A}) \subseteq \operatorname{rad}(\mathcal{A})$. Moreover, if $\mathcal{A}$ is primitive, then $F(x) = \alpha x$, $H(x) = \beta x$, for some $\alpha, \beta \in C_{\mathcal{A}}$ with $(\alpha\beta)^m = 1$ and $m = n$.

Theorem 3.3. *Let $\mathcal{A}$ be a prime Banach algebra, O_1, O_2 nonvoid open subsets on $\mathcal{A}$, $Q_{\mathcal{A}}$ its right Martindale quotient ring, $C_{\mathcal{A}}$ its extended centroid, $a \in C_{\mathcal{A}}$, F and H are two continuous generalized derivations of $\mathcal{A}$ associated with derivations d and h, respectively, such that*

$$aF(x)^m H(y)^m - x^n y^n = 0 \quad \forall\, (x, y) \in O_1 \times O_2$$

for two fixed positive integers $m \geqslant 1$ and $n \geqslant 1$. Then $F(x) = \alpha x$, $H(x) = \beta x$ for some $\alpha, \beta \in C_{\mathcal{A}}$. Moreover, $m = n$ and $a(\alpha\beta)^m = 1$.

P r o o f. By assumption

$$F(x)^m H(y)^m - x^n y^n = 0 \quad \forall\, (x, y) \in O_1 \times O_2. \tag{3.1}$$

Let $u \in \mathcal{A}$ and $x \in O_1$, then $x + tu \in O_1$ for a sufficiently small real t. F, H being continuous, one can obviously see that $F(ru) = rF(u)$ and $H(ru) = rH(u)$ for all $u \in A$, $r \in \mathbb{R}$. Taking $x + tu$ instead of x in equation (3.1), we get

$$Q(t) = a(F(x) + F(u)t)^m H(y)^m - (x + tu)^n y^n = 0. \tag{3.2}$$

Setting $Q(t) = \sum_{k=0}^{\max(m,n)} q_k(u,x,y)t^k$, if $m = n$, invoking Lemma 3.1, we obtain $q_k(u,x,y) = 0$ for all $k \in \{0,\dots,m\}$. In particular, $q_m(u,x,y) = 0$, thus

$$aF(u)^m H(y)^m - u^m y^m = 0 \quad \forall\,(u,y) \in \mathcal{A} \times O_2.$$

Similarly, by acting on y instead of x, one can easily get to

$$aF(u)^m H(v)^m - u^m v^m = 0 \quad \forall\,u,v \in \mathcal{A}.$$

By Theorem 2.1, we get the required results.

Suppose now that $m < n$, the right choice of the coefficient yields

$$p_n(u,x,y) = u^n y^n = 0 \quad \forall\,(u,y) \in \mathcal{A} \times O_2.$$

At the end, we get $u^n v^n = 0$ for all $u,v \in \mathcal{A}$. Substituting v by u and invoking [4], Lemma 1.1, it follows that $\mathcal{A}$ has a nonzero nilpotent ideal, absurd.

Now if $m > n$, it follows that $p_m(u,x,y) = aF(u)^m H(v)^m = 0$ for all $u,v \in \mathcal{A}$. The main equation leads to $u^n v^n = 0$ for all $u,v \in \mathcal{A}$ and we obtain the same contradiction. Then necessarily $m = n$. □

Theorem 3.4. *Let $\mathcal{A}$ be a prime Banach algebra, O_1, O_2 nonvoid open subsets on $\mathcal{A}$, $Q_{\mathcal{A}}$ its right Martindale quotient ring, $C_{\mathcal{A}}$ its extended centroid, F and H be two continuous generalized derivations of $\mathcal{A}$ associated with derivations d and h, respectively, such that*

$$(F(x) \circ H(y))^m - (x \circ y)^n = 0 \quad \forall\,(x,y) \in O_1 \times O_2$$

with $m \geqslant 1$ and $n \geqslant 1$ be two fixed positive integers. Then $F(x) = \alpha x$ and $H(x) = \beta x$ for some $\alpha, \beta \in C_{\mathcal{A}}$. Moreover, $m = n$ and $(\alpha\beta)^m = 1$.

Proof. Assume that

$$(F(x) \circ H(y))^m - (x \circ y)^n = 0 \quad \forall\,(x,y) \in O_1 \times O_2. \tag{3.3}$$

Let $u \in \mathcal{A}$. For a sufficiently small real s, one can replace x by $x+su$ in equation (3.3)

$$P(s) = (F(x) \circ H(y) + (F(u) \circ H(y))s)^m - (x \circ y + (u \circ y)s)^n = 0. \tag{3.4}$$

Set $P(s) = \sum_{k=0}^{\max(m,n)} p_k(u,x,y)s^k$. If $m = n$, a direct application of Lemma 3.1 leads to

$$p_m(u,x,y) = (F(u) \circ H(y))^m - (u \circ y)^m = 0 \quad \forall\,(u,y) \in \mathcal{A} \times O_2.$$

By adopting a similar approach, we get

$$(F(u)\circ H(v))^m-(u\circ v)^m=0\quad\forall\, u,v\in\mathcal{A}$$

and application of Theorem 2.2 gives the required conclusion.

Now if $m<n$, a suitable choice of the right coefficient yields

$$p_n(u,x,y)=(u\circ y)^n=0\quad\forall\,(u,y)\in\mathcal{A}\times O_2.$$

Then $(u\circ v)^n=0$ for all $u,v\in\mathcal{A}$. Replacing v by u and using [4], Lemma 1.1, we get a contradiction.

Regarding the case where $m>n$, we get $p_m(u,x,y)=(F(u)\circ H(v))^m=0$ for all $u,v\in\mathcal{A}$. The main equation becomes $(x\circ y)^n=0$ for all $(x,y)\in O_1\times O_2$. Arguing as in the last case, we obtain the same contradiction. Accordingly, $m=n$. $\square$

The following example shows that the primeness hypothesis in Theorems 2.1–2.2 is not superfluous.

Example 3.1. Let us consider the ring $R=\left\{\begin{pmatrix}0&x&y\\0&0&z\\0&0&0\end{pmatrix}\colon x,y,z\in\mathbb{Z}\right\}$ and $I=\left\{\begin{pmatrix}0&x&y\\0&0&0\\0&0&0\end{pmatrix}\colon x,y\in\mathbb{Z}\right\}$ an ideal of R. Define $F,H\colon R\to R$ with $F\begin{pmatrix}0&x&y\\0&0&z\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&yx\\0&0&0\\0&0&0\end{pmatrix}$ and $H\begin{pmatrix}0&x&y\\0&0&z\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&x\\0&0&z\\0&0&0\end{pmatrix}$. Obviously F and H are generalized derivations on R. Fix $a\in C\setminus\{0\}$. It is straightforward that $aF(X)^mH(Y)^m=X^nY^n$ and $(F(X)\circ H(Y))^m=(X\circ Y)^n$ for all $X,Y\in I$. However, conclusions of Theorems 2.1 and 2.2 are not satisfied.

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