TOTALLY CONTACT UMBILICAL SCREEN-SLANT AND SCREEN-TRANSVERSAL LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KENMOTSU MANIFOLD

Payel Karmakar, Kolkata

Received June 27, 2023. Published online April 17, 2024.
Communicated by Pavel Pyrih

Abstract. We study totally contact umbilical screen-slant lightlike submanifolds and totally contact umbilical screen-transversal lightlike submanifolds of an indefinite Kenmotsu manifold. We prove a characterization theorem of totally contact umbilical screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold. We further prove some results on a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold, such as the necessary and sufficient conditions for the screen distribution $S(TM)$ to be integrable and for the induced connection ∇ to be a metric connection.

Keywords: indefinite Kenmotsu manifold; lightlike submanifold; totally contact umbilical screen-slant lightlike submanifold; totally contact umbilical radical screen-transversal lightlike submanifold

MSC 2020: 53C15, 53C20, 53C25, 53C40, 53C50

1. Introduction

The general theory of lightlike submanifolds of a semi-Riemannian manifold was developed by Duggal and Bejancu in 1996 (see [3]). Later, Sahin characterized lightlike submanifolds in many ways. In 2006, he introduced the notion of transversal lightlike submanifolds and studied some differential geometric properties of those submanifolds (see [11]). In 2008, he initiated the study of screen transversal lightlike submanifolds (see [12]). Gupta introduced the notions of slant and screen slant submanifolds in indefinite Kenmotsu manifolds, respectively, in 2011 with Sharfuddin

The author has been sponsored by the University Grants Commission (UGC) Senior Research Fellowship, India, under grant UGC-Ref. No. 1139/(CSIR-UGC NET JUNE 2018).

DOI: 10.21136/MB.2024.0095-23

(see [7]) and in 2010 with Upadhyay (see [8]). Gupta and Sharfuddin also conceptualised screen transversal lightlike submanifolds in the context of indefinite cosymplectic manifolds in 2010 (see [5]) and later in the context of indefinite Kenmotsu manifolds in 2011 (see [6]). In 2012, Haider et al. in [10] studied totally contact umbilical screen transversal lightlike submanifolds of an indefinite Sasakian manifold and recently, in 2021, Yadav et al. investigated the existence of totally contact umbilical screen-slant lightlike submanifolds of indefinite Sasakian manifolds (see [13]).

Motivated by the works mentioned above, in this paper we study totally contact umbilical screen-slant lightlike submanifolds and totally contact umbilical screen-transversal lightlike submanifolds of indefinite Kenmotsu manifold. This paper is divided into five sections. After introduction (first section) and preliminaries (second section), in the third section, we prove some results regarding screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold. In the fourth section, we prove a characterization theorem of totally contact umbilical screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold. In the last, i.e., the fifth section, we further prove some results on a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold, such as the necessary and sufficient conditions for the screen distribution $S(TM)$ to be integrable and for the induced connection ∇ to be a metric connection.

2. Preliminaries

A submanifold (M^m, g) which is immersed in a proper semi-Riemannian manifold $(\widetilde{M}^{m+n}, \widetilde{g})$ is called a *lightlike submanifold* (see [3]) if the metric g induced from $\widetilde{g}$ is degenerate and the radical distribution $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$ is of rank r such that $1 \leqslant r \leqslant m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, i.e.,

$$TM = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S(TM).$$

Let us consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $\mathrm{Rad}(TM)$ in $TM^{\perp}$, i.e.,

$$TM^{\perp} = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S(TM^{\perp}).$$

Since for any local basis $\{\xi_i\}$ of $\mathrm{Rad}(TM)$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $S(TM)^{\perp}$ such that $\widetilde{g}(\xi_i, N_j) = \delta_{ij}$ and $\widetilde{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $\operatorname{ltr}(TM)$ locally spanned by $\{N_i\}$. Let $\operatorname{tr}(TM)$ be the complementary (not orthogonal) vector bundle to TM in $T\widetilde{M}$. Now we have the following decompositions (see [3]):

$$T\widetilde{M}|_M = TM \oplus \operatorname{tr}(TM), \quad \operatorname{tr}(TM) = S(TM^{\perp}) \oplus_{\mathrm{orth}} \operatorname{ltr}(TM),$$
$$T\widetilde{M}|_M = S(TM) \oplus_{\mathrm{orth}} [\operatorname{Rad}(TM) \oplus \operatorname{ltr}(TM)] \oplus_{\mathrm{orth}} S(TM^{\perp}).$$

A submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\widetilde{M}$ is called

▷ *r-lightlike* if $r < \min\{m, n\}$,
▷ *co-isotropic* if $r = n < m$, $S(TM^{\perp}) = \{0\}$,
▷ *isotropic* if $r = m < n$, $S(TM) = \{0\}$,
▷ *totally lightlike* if $r = m = n$, $S(TM) = \{0\} = S(TM^{\perp})$.

An odd dimensional semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is called an *indefinite almost contact metric manifold* (see [1]) if it admits an indefinite almost contact structure (φ, ξ, η), where φ is a tensor field of type $(1,1)$, ξ is a vector field and η is a 1-form satisfying for all $X, Y \in \chi(\widetilde{M})$

$$\widetilde{g}(\varphi X, \varphi Y) = \widetilde{g}(X, Y) - \varepsilon\eta(X)\eta(Y), \quad \widetilde{g}(\xi, \xi) = \varepsilon = \pm 1, \tag{2.1}$$
$$\varphi^2 X = -X + \eta(X)\xi, \quad \widetilde{g}(X, \xi) = \varepsilon\eta(X), \tag{2.2}$$
$$\widetilde{g}(X, \varphi Y) = -\widetilde{g}(\varphi X, Y), \tag{2.3}$$
$$\eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \eta(\xi) = 1. \tag{2.4}$$

De and Sarkar in [2] introduced the notion of ε-Kenmotsu manifolds with indefinite metric. An *indefinite Kenmotsu manifold* $\widetilde{M}(\varphi, \xi, \eta, \widetilde{g})$ satisfies the following structure equations for all $X, Y \in \chi(\widetilde{M})$:

$$(\widetilde{\nabla}_X \varphi)Y = \widetilde{g}(\varphi X, Y)\xi - \varepsilon\eta(Y)\varphi X, \tag{2.5}$$
$$\widetilde{\nabla}_X \xi = \varepsilon[X - \eta(X)\xi], \tag{2.6}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\widetilde{g}$.

A lightlike submanifold M of an indefinite Kenmotsu manifold $\widetilde{M}$, with the structure vector field ξ tangent to M, is called a *totally contact umbilical lightlike submanifold* (see [14]) if for a vector field α transversal to M and for all $X, Y \in \Gamma(TM)$,

$$h(X, Y) = [g(X, Y) - \eta(X)\eta(Y)]\alpha + \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \tag{2.7}$$

where h is a symmetric bilinear form on $\Gamma(TM)$ with values in $\Gamma(\operatorname{tr}(TM))$ known as the *second fundamental form*. If $\alpha = 0$, then M is called a *totally contact geodesic lightlike submanifold.*

Now, equating components of (2.7) belonging to $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$, respectively, we have (see [4])

$$h^l(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha_l + \eta(X)h^l(Y,\xi) + \eta(Y)h^l(X,\xi), \tag{2.8}$$

$$h^s(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha_s + \eta(X)h^s(Y,\xi) + \eta(Y)h^s(X,\xi), \tag{2.9}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$ (L, S are the projection morphisms of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$, $S(TM^{\perp})$, respectively) and $\alpha_l \in \Gamma(\mathrm{ltr}(TM))$, $\alpha_s \in \Gamma(S(TM^{\perp}))$. h^l and h^s are called the *lightlike second fundamental form* and the *screen second fundamental form* of M, respectively.

Let M be a lightlike submanifold of an indefinite Kenmotsu manifold $\widetilde{M}$ and ∇, $\widetilde{\nabla}$ be the Levi-Civita connections on M, $\widetilde{M}$, respectively. The Gauss and Weingarten formulae are given by:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \forall\, X,Y \in \Gamma(TM), \tag{2.10}$$

$$\widetilde{\nabla}_X V = -A_V X + \nabla^t_X V \quad \forall\, X \in \Gamma(TM),\ V \in \Gamma(\mathrm{tr}(TM)), \tag{2.11}$$

where $\nabla_X Y$, $A_V X \in \Gamma(TM)$ and $h(X,Y)$, $\nabla^t_X V \in \Gamma(\mathrm{tr}(TM))$. Here A is a linear operator on TM known as the *shape operator* and ∇^t is a linear connection on $\mathrm{tr}(TM)$ known as the *transversal linear connection* on M.

Now, the equations (2.10) and (2.11) further reduce to

$$\begin{gathered} \widetilde{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y) \quad \forall\, X,Y \in \Gamma(TM), \\ \widetilde{\nabla}_X V = -A_V X + D^l(X,V) + D^s(X,V) \quad \forall\, X \in \Gamma(TM),\ V \in \Gamma(\mathrm{tr}(TM)), \end{gathered} \tag{2.12}$$

where $D^l(X,V) = L(\nabla^t_X V)$, $D^s(X,V) = S(\nabla^t_X V)$.

In particular, we have

$$\widetilde{\nabla}_X U = -A_U X + \nabla^l_X U + D^s(X,U) \quad \forall\, U \in \Gamma(\mathrm{ltr}(TM)), \tag{2.13}$$

$$\widetilde{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W) \quad \forall\, W \in \Gamma(S(TM^{\perp})), \tag{2.14}$$

where ∇^l and ∇^s are linear connections on $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$ called the *lightlike transversal connection* and the *screen transversal connection* on M, respectively.

Again, from (2.12)–(2.14) we get

$$\widetilde{g}(h^s(X,Y),W) + \widetilde{g}(Y,D^l(X,W)) = g(A_W X,Y), \tag{2.15}$$

$$\widetilde{g}(D^s(X,U),W) = \widetilde{g}(U,A_W X). \tag{2.16}$$

Let $\overline{P}$ be the projection morphism of TM on $S(TM)$, then we have for all $X, Y \in \Gamma(TM)$, $V \in \Gamma(\mathrm{Rad}(TM))$,

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \tag{2.17}$$

$$\nabla_X V = -A^*_V X + \nabla^{*t}_X V, \tag{2.18}$$

where h^* is the local second fundamental form on $S(TM)$ and A^* is the shape operator of $\mathrm{Rad}(TM)$, $\nabla^*_X \overline{P}Y, A^*_V X \in \Gamma(S(TM))$ and $h^*(X, \overline{P}Y), \nabla^{*t}_X V \in \Gamma(\mathrm{Rad}(TM))$. Here ∇^* and ∇^{*t} are induced connections on $S(TM)$ and $\mathrm{Rad}(TM)$, respectively.

3. Screen-slant lightlike submanifolds

In this section, we prove some results regarding screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold.

Let M be a $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $\widetilde{M}$ of index $2q$ such that $2q < \dim(M)$ with structure vector field ξ tangent to M, then M is called a *screen-slant lightlike submanifold* of $\widetilde{M}$ if the following conditions are satisfied (see [8]):

(i) $\mathrm{Rad}(TM)$ is invariant with respect to φ, i.e., $\varphi(\mathrm{Rad}(TM)) \subseteq \mathrm{Rad}(TM)$,

(ii) for any nonzero vector field Y tangent to $S(TM) = D \oplus_{\mathrm{orth}} \langle\xi\rangle$ at $y \in M$, the angle $\theta(Y)$ (known as the *slant angle*) between φY and $S(TM)$ is constant, where D is the complementary distribution to $\langle\xi\rangle$ in $S(TM)$ and Y, ξ are linearly independent.

M is called *proper* if $D \neq \{0\}$, $\theta \neq 0, \frac{1}{2}\pi$, and is called a *screen real lightlike submanifold* if $\theta = \frac{1}{2}\pi$. Then we have the decomposition

$$TM = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} D \oplus_{\mathrm{orth}} \langle\xi\rangle .$$

Let P, Q be the projection morphisms of TM on $\mathrm{Rad}(TM)$, D, respectively, then for any $X \in \Gamma(TM)$, we have

$$X = PX + QX + \eta(X)\xi, \tag{3.1}$$

where $PX \in \Gamma(\mathrm{Rad}(TM))$, $QX \in \Gamma(D)$.

Again, for any $X \in \Gamma(TM)$, we have

$$\varphi X = TX + \omega X, \tag{3.2}$$

where $TX \in \Gamma(TM)$ and $\omega X \in \Gamma(\mathrm{tr}(TM))$ are the tangential and transversal components of φX, respectively.

Now, applying φ on (3.1) we get

$$\varphi X = TPX + TQX + \omega QX. \tag{3.3}$$

$S(TM^{\perp})$ can be decomposed as

$$S(TM^{\perp}) = \omega Q(S(TM)) \oplus_{\text{orth}} \mu,$$

where μ is an invariant subspace of $T\widetilde{M}$. Then for any $W \in \Gamma(S(TM^{\perp}))$, we have

$$\varphi W = BW + CW, \tag{3.4}$$

where $BW \in \Gamma(S(TM))$, $CW \in \Gamma(S(TM^{\perp}))$.

Also, for any $N \in \Gamma(\text{ltr}(TM))$,

$$\varphi N = CN, \tag{3.5}$$

where $CN \in \Gamma(\text{ltr}(TM))$.

Now, we state and prove some results.

Theorem 3.1. *Let M be a $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $\widetilde{M}$ with constant index $2q < \dim(M)$, then M is a screen-slant lightlike submanifold if and only if there exists a constant $\lambda \in [-1, 0]$ such that for all $X \in \Gamma(S(TM))$,*

$$(P \circ T)^2 X = \lambda[-X + \eta(X)\xi], \tag{3.6}$$

where $\lambda = \cos^2 \theta|_{S(TM)}$.

P r o o f. The proof follows from Theorem 3.1 in [9]. □

Corollary 3.2. *Let (M, g) be a screen-slant lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M}, \widetilde{g})$, then for all $X, Y \in \Gamma(TM)$,*

$$g(TQX, TQY) = \cos^2 \theta|_{S(TM)}[g(X, Y) - \varepsilon\eta(X)\eta(Y)], \tag{3.7}$$

$$\widetilde{g}(\omega QX, \omega QY) = \sin^2 \theta|_{S(TM)}[g(X, Y) - \varepsilon\eta(X)\eta(Y)]. \tag{3.8}$$

P r o o f. The proof follows from Corollary 3.2 in [9]. □

Theorem 3.3. *Let (M, g) be a screen-slant lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M}, \widetilde{g})$, then for all $X, Y \in \Gamma(TM)$,*

$$(\nabla_X T)Y = A_{\omega Y}X + Bh^s(X, Y) + \widetilde{g}(\varphi X, Y)\xi - \varepsilon\eta(Y)TX, \tag{3.9}$$

$$\begin{aligned}(\nabla_X \omega)Y = {} & Ch^s(X, Y) + Ch^l(X, Y) - h^s(X, TY) - h^l(X, TY) \\ & - D^l(X, \omega Y) - \varepsilon\eta(Y)\omega X,\end{aligned} \tag{3.10}$$

where $(\nabla_X T)Y = \nabla_X TY - T(\nabla_X Y)$ and $(\nabla_X \omega)Y = \nabla^s_X \omega Y - \omega(\nabla_X Y)$.

P r o o f. From (2.5) we get

$$\widetilde{\nabla}_X \varphi Y = \varphi \widetilde{\nabla}_X Y + \widetilde{g}(\varphi X, Y)\xi - \varepsilon\eta(Y)\varphi X. \tag{3.11}$$

Applying (3.2) on (3.11) we obtain

$$\widetilde{\nabla}_X (TY + \omega Y) = \varphi \widetilde{\nabla}_X Y + \widetilde{g}(\varphi X, Y)\xi - \varepsilon\eta(Y)(TX + \omega X),$$

on which applying (2.12), (2.14), (3.2), (3.4), (3.5), we get

$$\begin{aligned} \nabla_X TY + h^l(X, TY) + h^s(X, TY) - A_{\omega Y} X + \nabla^s_X \omega Y + D^l(X, \omega Y) \\ = T\nabla_X Y + \omega \nabla_X Y + Ch^l(X, Y) + Bh^s(X, Y) + Ch^s(X, Y) \\ + \widetilde{g}(\varphi X, Y)\xi - \varepsilon\eta(Y)(TX + \omega X). \end{aligned} \tag{3.12}$$

Equating tangential and transversal components of (3.12) we obtain (3.9) and (3.10), respectively. □

4. Totally contact umbilical screen-slant lightlike submanifolds

In this section, we prove the following characterization theorem of totally contact umbilical screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold.

Theorem 4.1. *Let (M, g) be a totally contact umbilical screen-slant lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M}, \widetilde{g})$, then at least one of the following statements is true:*

(i) *M is a screen real lightlike submanifold,*
(ii) *$D = \{0\}$,*
(iii) *if M is a proper screen-slant lightlike submanifold, then $\alpha_s \in \Gamma(\mu)$.*

P r o o f. For any $Y = QY \in \Gamma(D)$, from (2.7) we have

$$h(TQY, TQY) = g(TQY, TQY)\alpha,$$

on which applying (2.3), (2.5), (2.10), (2.12), (2.14), (3.1), (3.2), (3.7), we get

$$\begin{aligned} \varphi(\nabla_{TQY} QY + h^l(TQY, QY) + h^s(TQY, QY)) + A_{\omega QY} TQY - \nabla^s_{TQY} \omega QY \\ - D^l(TQY, \omega QY) - \nabla_{TQY} TQY - g(TQY, TQY)\xi = \cos^2\theta g(Y, Y)\alpha, \end{aligned}$$

which (by the help of (2.8), (2.9), (3.2)) reduces to

$$
\begin{aligned}
(4.1)\qquad T\nabla_{TQY}QY + \omega\nabla_{TQY}QY + A_{\omega QY}TQY - \nabla^s_{TQY}\omega QY - D^l(TQY, \omega QY)\\
- \nabla_{TQY}TQY - g(TQY, TQY)\xi = \cos^2\theta g(Y,Y)\alpha,
\end{aligned}
$$

since $g(TQY, QY) = \widetilde{g}(\varphi Y, Y) = -\widetilde{g}(Y, \varphi Y) = -g(TQY, QY) \Rightarrow g(TQY, QY) = 0$.

Equating transversal components of (4.1) we obtain

$$
(4.2)\qquad \omega\nabla_{TQY}QY - \nabla^s_{TQY}\omega QY - D^l(TQY, \omega QY) = \cos^2\theta g(Y,Y)\alpha.
$$

Now, taking covariant derivative of (3.8) with respect to TQY we get

$$
(4.3)\qquad \widetilde{g}(\nabla^s_{TQY}\omega QY, \omega QY) = \sin^2\theta g(\nabla^s_{TQY}Y, Y).
$$

Again, from (3.8) we have

$$
(4.4)\qquad \widetilde{g}(\omega\nabla_{TQY}QY, \omega QY) = \sin^2\theta g(\nabla^s_{TQY}Y, Y).
$$

Now, taking inner product of (4.2) with ωQY we obtain

$$
\begin{aligned}
\widetilde{g}(\omega\nabla_{TQY}QY, \omega QY) - \widetilde{g}(\nabla^s_{TQY}\omega QY, \omega QY) &= \cos^2\theta g(Y,Y)\widetilde{g}(\alpha_s, \omega QY)\\
&\Rightarrow \cos^2\theta g(Y,Y)\widetilde{g}(\alpha_s, \omega QY) = 0 \text{ (by (4.3), (4.4))}\\
&\Rightarrow \theta = \frac{\pi}{2} \text{ or } Y = 0 \text{ or } \alpha_s \in \Gamma(\mu),
\end{aligned}
$$

which gives that either M is a screen real lightlike submanifold or $D = \{0\}$ or $\alpha_s \in \Gamma(\mu)$ if M is proper. This completes the proof. □

5. Totally contact umbilical radical screen-transversal lightlike submanifolds

In this section, we prove some results on a totally contact umbilical radical screen-transversal lightlike submanifold M of an indefinite Kenmotsu manifold $\widetilde{M}$, such as the necessary and sufficient conditions for the screen distribution $S(TM)$ to be integrable and for the induced connection ∇ to be a metric connection.

First we state the following definitions from [12].

▷ An r-lightlike submanifold M of an indefinite Kenmotsu manifold $\widetilde{M}$ is called a *screen-transversal lightlike submanifold* if $\varphi(\mathrm{Rad}(TM)) \subseteq S(TM^\perp)$.

▷ A screen-transversal lightlike submanifold M of an indefinite Kenmotsu manifold $\widetilde{M}$ is called a *radical screen-transversal lightlike submanifold* if $S(TM)$ is invariant with respect to φ, i.e., $\varphi(S(TM)) \subseteq S(TM)$.

Next, we prove the following results.

Theorem 5.1. *Let (M,g) be a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M},\widetilde{g})$, then $S(TM)$ is integrable if and only if α_s has no component in $\varphi(\mathrm{Rad}(TM))$.*

Proof. For any $X,Y\in\Gamma(S(TM))$ and $N\in\Gamma(\mathrm{Rad}(TM))$, using (2.1), (2.3), (2.5), (2.9), (2.12) we get

$$\widetilde{g}([X,Y],N)=\widetilde{g}(h^s(X,\varphi Y)-h^s(Y,\varphi X),\varphi N)=2g(X,\varphi Y)\widetilde{g}(\alpha_s,\varphi N),$$

which implies that $[X,Y]\in\Gamma(S(TM))$ for all $X,Y\in\Gamma(S(TM))$ if and only if $\widetilde{g}(\alpha_s,\varphi N)=0$ for all $N\in\Gamma(\mathrm{Rad}(TM))$.

This completes the proof. □

Theorem 5.2. *Let (M,g) be a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M},\widetilde{g})$, then $h^*=0$ if and only if α_s has no component in $\varphi(\mathrm{Rad}(TM))$.*

Proof. For any $X,Y\in\Gamma(S(TM))$, using (2.5), (2.12) we have

$$\begin{aligned}\nabla_X\varphi Y+h^l(X,\varphi Y)+h^s(X,\varphi Y)\\=\widetilde{g}(\varphi X,Y)\xi-\varepsilon\eta(Y)\varphi X+\varphi(\nabla_XY+h^l(X,Y)+h^s(X,Y)).\end{aligned}\tag{5.1}$$

Taking inner product of (5.1) with φN for any $N\in\Gamma(\mathrm{Rad}(TM))$, we obtain

$$\widetilde{g}(h^s(X,\varphi Y),\varphi N)=\widetilde{g}(\varphi\nabla_XY,\varphi N).\tag{5.2}$$

Now, using (2.1), (2.9), (2.17) in (5.2) we get

$$\widetilde{g}(\alpha_s,\varphi N)g(X,\varphi Y)=\widetilde{g}(h^*(X,Y),N),$$

which implies our assertion. □

Theorem 5.3. *Let (M,g) be a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M},\widetilde{g})$, then the induced connection ∇ on M is a metric connection if and only if α_s has no component in $\varphi(\mathrm{Rad}(TM))$.*

P r o o f. For any $X \in \Gamma(TM)$ and $N \in \Gamma(\mathrm{Rad}(TM))$, using (2.5) we get

$$\widetilde{\nabla}_X \varphi N - \varphi(\widetilde{\nabla}_X N) = \widetilde{g}(\varphi X, N)\xi,$$

on which applying φ and then using (2.2), (2.4), we obtain

$$\widetilde{\nabla}_X N = -\varphi(\widetilde{\nabla}_X \varphi N). \tag{5.3}$$

Using (2.12), (2.14) in (5.3) and then taking inner product with $Y \in \Gamma(S(TM))$ and then using (2.3) we get

$$g(\nabla_X N, Y) = -g(A_{\varphi N}X, \varphi Y) + \widetilde{g}(\nabla^s_X \varphi N, \varphi Y) + \widetilde{g}(D^l(X, \varphi N), \varphi Y),$$

in which using (2.9), (2.15), we obtain

$$g(\nabla_X N, Y) = -g(X, \varphi Y)\widetilde{g}(\alpha_s, \varphi N).$$

Therefore, ∇ is a metric connection on M if and only if $\mathrm{Rad}(TM)$ is parallel if and only if $\nabla_X N \in \Gamma(\mathrm{Rad}(TM))$ for all $X \in \Gamma(TM)$, $N \in \Gamma(\mathrm{Rad}(TM))$ if and only if $\widetilde{g}(\alpha_s, \varphi N) = 0$ for all $N \in \Gamma(\mathrm{Rad}(TM))$. This completes the proof. □

Theorem 5.4. *Let (M, g) be a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M}, \widetilde{g})$, then*

(i) $A_{\varphi N}X = [X - \varepsilon\eta(X)\xi]\widetilde{g}(\alpha_s, \varphi N) + \varepsilon\eta(X)\varphi N + D^l(X, \varphi N)$ *for all* $X \in \Gamma(S(TM)), N \in \Gamma(\mathrm{ltr}(TM))$,

(ii) $A_{\varphi N}X = X\widetilde{g}(\alpha_s, \varphi N) + D^l(X, \varphi N)$ *for all* $X \in \Gamma(\mathrm{Rad}(TM))$, $N \in \Gamma(\mathrm{ltr}(TM))$.

P r o o f. Replacing W by φN in (2.15) we have

$$g(A_{\varphi N}X, Y) = \widetilde{g}(h^s(X, Y), \varphi N) + \widetilde{g}(Y, D^l(X, \varphi N)),$$

on which applying (2.6), (2.9), (2.12), we get

$$\begin{aligned} g(A_{\varphi N}X, Y) = {} & [g(X, Y) - \eta(X)\eta(Y)]\widetilde{g}(\alpha_s, \varphi N) + \varepsilon\eta(X)\widetilde{g}(Y, \varphi N) \\ & + \varepsilon\eta(Y)\widetilde{g}(X, \varphi N) + \widetilde{g}(Y, D^l(X, \varphi N)), \end{aligned}$$

which gives

$$A_{\varphi N}X = [X - \varepsilon\eta(X)\xi]\widetilde{g}(\alpha_s, \varphi N) + \varepsilon\eta(X)\varphi N + \widetilde{g}(X, \varphi N)\xi + D^l(X, \varphi N). \tag{5.4}$$

Then (i) and (ii) immediately follow from (5.4) restricting X to $S(TM)$ and $\mathrm{Rad}(TM)$, respectively. □

A c k n o w l e d g e m e n t. The author is thankful to the referee for valuable suggestions leading to improving the quality of the paper.

References

[1] *A. Bonome, R. Castro, E. García-Río, L. Hervella*: Curvature of indefinite almost contact manifolds. J. Geom. *58* (1997), 66–86. zbl MR doi

[2] *U. C. De, A. Sarkar*: On (ε)-Kenmotsu manifolds. Hadronic J. *32* (2009), 231–242. zbl MR

[3] *K. L. Duggal, A. Bejancu*: Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. Mathematics and Its Applications 364. Kluwer Academic, Dordrecht, 1996. zbl MR doi

[4] *K. L. Duggal, B. Sahin*: Lightlike submanifolds of indefinite Sasakian manifolds. Int. J. Math. Math. Sci. *2007* (2007), Article ID 57585, 21 pages. zbl MR doi

[5] *R. S. Gupta, A. Sharfuddin*: Screen transversal lightlike submanifolds of indefinite cosymplectic manifolds. Rend. Semin. Mat. Univ. Padova *124* (2010), 145–156. zbl MR doi

[6] *R. S. Gupta, A. Sharfuddin*: Screen transversal lightlike submanifolds of indefinite Kenmotsu manifolds. Sarajevo J. Math. *7* (2011), 103–113. zbl MR

[7] *R. S. Gupta, A. Sharfuddin*: Slant lightlike submanifolds of indefinite Kenmotsu manifolds. Turk. J. Math. *35* (2011), 115–127. zbl MR

[8] *R. S. Gupta, A. Upadhyay*: Screen slant lightlike submanifolds of indefinite Kenmotsu manifolds. Kyungpook Math. J. *50* (2010), 267–279. zbl MR doi

[9] *S. M. K. Haider, Advin, M. Thakur*: Screen slant lightlike submanifolds of indefinite Sasakian manifolds. Kyungpook Math. J. *52* (2012), 443–457. zbl MR doi

[10] *S. M. K. Haider, M. Thakur, Advin*: Totally contact umbilical screen transversal lightlike submanifolds of an indefinite Sasakian manifold. Note Mat. *32* (2012), 123–134. zbl MR doi

[11] *B. Sahin*: Transversal lightlike submanifolds of indefinite Kähler manifolds. An. Univ. Vest Timiş., Ser. Mat.-Inform. *44* (2006), 119–145. zbl MR

[12] *B. Sahin*: Screen transversal lightlike submanifolds of indefinite Kähler manifolds. Chaos Solitons Fractals *38* (2008), 1439–1448. zbl MR doi

[13] *A. Yadav, G. Shanker, R. Kaur*: An investigation on the existence of totally contact umbilical screen-slant lightlike submanifolds of indefinite Sasakian manifolds. Differ. Geom. Dyn. Syst. *23* (2021), 244–254. zbl MR

[14] *K. Yano, M. Kon*: Structures on Manifolds. Series in Pure Mathematics 3. World Scientific, Singapore, 1984. zbl MR doi

Author's address: *Payel Karmakar*, Department of Mathematics, Jadavpur University, 188, Raja Subodh Chandra Mallick Road, Kolkata-700032, West Bengal, India, e-mail: `payelkarmakar632@gmail.com`.